General Solution to the Logistic Differential Equation

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Let us be concerned with a population P that varies in direct proportion to its current quantity and its maximum quantity $P_{\max}$ with constant of proportionality k; that is,

$$\frac{dP}{dt} = kP\left(P_{\max} - P\right)$$

Separation of variables gives

$$\frac{dP}{P\left(P_{\max} - P\right)} = k\,dt$$

of which the left side can be integrated using partial fractions (for some constants A and B) as follows:

$$\int \left(\frac{A}{P} + \frac{B}{P_{\max} - P} \right) dP = \int k\,dt$$

so we have

$$\frac{A}{P} + \frac{B}{P_{\max} - P} = \frac{1}{P(P_{\max} - P)}$$

and, clearing the denominators, $A\left(P_{\max} - P\right) + BP = 1$ which can be rearranged to $P(B - A) + P_{\max}A = 1$, so $P_{\max}A = 1$ and $B - A = 0$, which gives $A = B = \frac{1}{P_{\max}}$.

Therefore

$$\int \left(\frac{1/P_{\max}}{P} + \frac{1/P_{\max}}{P_{\max} - P} \right) dP = \int k\,dt$$

or

$$\frac{\ln P}{P_{\max}} + \frac{\ln\left(P_{\max} - P\right)}{P_{\max}} = kt + C$$

(noting that since we have two equivalent integrals, we need only include a constant of integration on one side). The second natural logarithm may be rewritten as $-\ln\left(P - P_{\max}\right)$, so we have

$$\frac{1}{P_{\max}} \left(\ln P - \ln\left(P - P_{\max}\right) \right) = kt + C$$

or

$$\frac{1}{P_{\max}} \left(\ln \frac{P}{P - P_{\max}} \right) = kt + C$$

We rearrange this to give

$$\ln \frac{P}{P - P_{\max}} = P_{\max}kt + C$$

for a different C, but a constant nonetheless; now, we must solve for P:

$$\frac{P}{P - P_{\max}} = Ce^{P_{\max}kt}$$

again for a different C. Now $P = (P - P_{\max}) Ce^{P_{\max}kt} = PCe^{P_{\max}kt} - P_{\max}Ce^{P_{\max}kt}$ or $P - PCe^{P_{\max}kt} = -P_{\max}Ce^{P_{\max}kt}$; equivalently, $P\left(1 - Ce^{P_{\max}kt}\right) = -P_{\max}Ce^{P_{\max}kt}$ and finally,

$$P = -\frac{P_{\max}Ce^{P_{\max}kt}}{1 - Ce^{P_{\max}kt}}$$

or, if you prefer,

$$P = \frac{P_{\max}Ce^{P_{\max}kt}}{Ce^{P_{\max}kt} - 1}$$

It can be shown that if the population at time $t = 0$ is C_0, then

$$C = \frac{P_0}{P_0 + P_{\max}}$$

so the equation is

$$P = \frac{P_{\max}\left(\frac{P_0}{P_0+P_{\max}}\right)e^{P_{\max}kt}}{\left(\frac{P_0}{P_0+P_{\max}}\right)e^{kt} - 1}$$

which can also be written as

$$P = \frac{P_0 P_{\max}e^{P_{\max}kt}}{P_0e^{kt} - P_0 - P_{\max}}$$