AP Calculus BC
Lesson 11.4 Polar Calculus

1. Consider the polar function $r(\theta) = 4\cos\theta$.

   a) Sketch a graph of the function:

   b) Find $\dfrac{dr}{d\theta}$

   c) Find the values of $\dfrac{dr}{d\theta}$ at the points where $\theta = \frac{\pi}{6}$ and $\theta = \frac{3\pi}{4}$.
   What do these values mean in terms of the motion along the graph?

   d) When is $\dfrac{dr}{d\theta}$ positive? negative? zero?

   e) Find parametric equations for the function, using θ as the parameter.

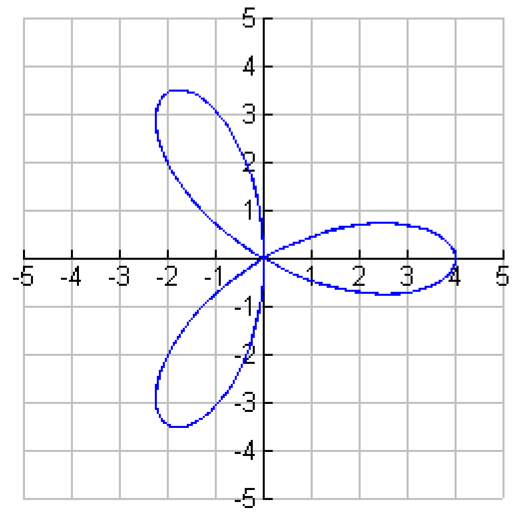
   f) Find $\dfrac{dy}{dx}$ in terms of θ. Simplify, if you can.

   g) Find an equation for the line tangent to the graph at the point where $\theta = \frac{\pi}{6}$.

   h) Find $\dfrac{d^2y}{dx^2}$ in terms of θ, and evaluate at $\theta = \frac{\pi}{6}$.

2. Consider the function $r(\theta) = 4\cos(3\theta)$:

a) Find a range of values for θ that would trace the rightmost loop of the graph exactly once.



b) Identify any points on the graph where the derivative $\dfrac{dr}{d\theta}$ is 0.

c) For what value(s) of θ is $\dfrac{dr}{d\theta}$ positive?

d) Find the point(s) on the graph where the tangent line is horizontal.

e) Use Calculus to find the leftmost point(s) on the graph.

3. Given the parametric equations for a polar function $r = f(\theta)$: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, find an expression for $\dfrac{dy}{dx}$ in terms of r and θ.

4. Show that $\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2 + r^2$, given that $x = r\cos\theta$ and $y = r\sin\theta$.

[Hint: treat r as a function of θ and use the chain rule to differentiate]

5. Find the total length of the following polar curves:

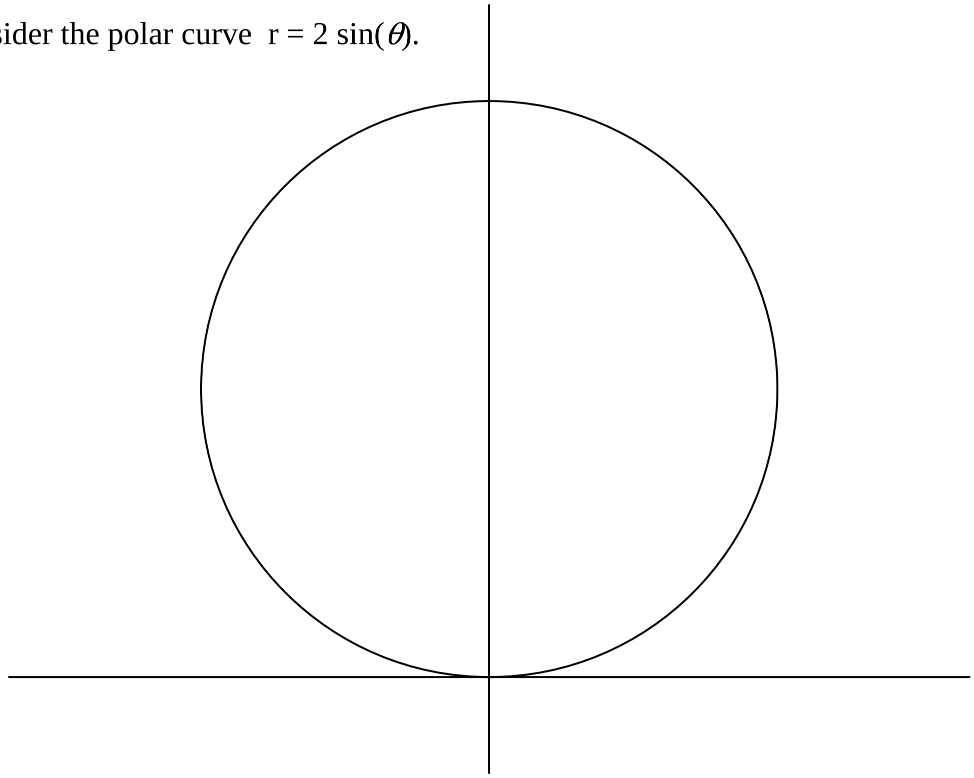
(a) $r = 4 - 8\cos(\theta)$

(b) $r = \sin(3\theta)$

(c) $r = e^{\theta}$ for $0 \le \theta \le 2$

6. (a) Find (or recall) a formula for the area of a sector of a circle in terms of the radius of the circle, r, and the central angle of the sector, θ.

(b) Consider the polar curve $r = 2\sin(\theta)$.



Find the area enclosed by this curve using the following method:

i. Divide the area into six parts using the lines

$$\theta = \frac{\pi}{6},\ \theta = \frac{\pi}{3},\ \theta = \frac{\pi}{2},\ \theta = \frac{2\pi}{3},\ \theta = \frac{5\pi}{6}.$$

ii. Approximate the area of each part using circular sectors.
Use the largest angle measure in each part to determine the radius of the sector.

iii. Write a Riemann sum that would approximate the area inside the curve for n divisions.

iv. Write an integral that represents the exact area, and evaluate the integral.

7. Draw and shade the area represented by the integral:

a) $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{2}(4\cos\theta)^2 \, d\theta$

b) $\int_{0}^{\frac{\pi}{4}} \frac{1}{2}(4\sin\theta)^2 \, d\theta$

c) $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{2}(4\cos(3\theta))^2 \, d\theta$

8. [1993-BC5] Consider the polar curve $r = 2\sin(3\theta)$, for $0 \le \theta \le \pi$.

(a) In the *xy*-plane sketch the curve.

(b) Find the area of the region inside the curve.

(c) Find the slope of the curve at the point where $\theta = \frac{\pi}{4}$.

9. Find the following areas

(a) The area inside one leaf of the rose $r = 4\cos(2\theta)$

(b) The area inside the outer loop but outside the inner loop of $r = 4 - 8\cos(\theta)$.

10. [1990 BC 4] Let R be the region inside the graph of the polar curve $r = 2$ and outside the graph of the polar curve $r = 2(1 - \sin(\theta))$.

(a) Sketch the two curves in the xy-plane.

(b) Find the area of R.

11. [1984 BC 4] Consider the curves $r = 3\cos(\theta)$ and $r = 1 + \cos(\theta)$.

(a) Sketch the two curves in the xy-plane.

(b) Find the area of the region inside the curve $r = 3\cos(\theta)$ and outside the curve $r = 1 + \cos(\theta)$.

12. Find the area shared by the circles $r = 2\cos(\theta)$ and $r = 2\sin(\theta)$.