Some Short Sequence and Series Strategies

Squeeze Theorem

If $a_n \le b_n \le c_n$ for $n \ge n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Geometric Sequences

The sequence $\{r^n\}$ is convergent if $-1 < r \le 1$ and divergent for all other values of r.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & if \;\; -1 < r < 1 \\ 1 & if \;\; r = 1 \end{cases}$$

Definition of Convergence for an Infinite Series

When $S_n = \sum_{k=1}^{n} a_k$, if $\lim_{n\to\infty} S_n = S$, for some finite number *S*, then the series $\sum_{k=1}^{\infty} a_k$ converges to the limit *S*. Otherwise the series diverges.

Geometric Series

If $|r| < 1$, the geometric series $\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \ldots$ converges to $\dfrac{a}{1-r}$.
If $a \ne 0$ and $|r| \ge 1$, the series diverges.

(n^{th} Term) Test for Divergence

If $\lim_{n\to\infty} a_n \ne 0$, then $\sum_{k=1}^{\infty} a_k$ diverges.

Integral Test for Positive Series

Suppose that, for $x \ge 1$, the function *a(x)* is continuous, positive and decreasing. Consider the series $\sum_{k=1}^{\infty} a_k$ and the integral $\int_1^{\infty} a(x)dx$.

- If either diverges, so does the other.
- If either converges, so does the other. In this case,

$$\int_1^{\infty} a(x)dx \le \sum_{k=1}^{\infty} a_k \le a_1 + \int_1^{\infty} a(x)dx$$

- If the series converges, then

$$\int_{n+1}^{\infty} a(x)dx \le R_n = \sum_{k=n+1}^{\infty} a_k \le \int_n^{\infty} a(x)dx.$$

p-Series (hyperharmonic) Test

The p-series $\sum_{k=1}^{\infty} \dfrac{1}{k^p}$ converges *if and only if* $p > 1$

Comparison Test for Nonnegative Series

Suppose that for $k \geq 1$, $0 \leq a_k \leq b_k$. Consider the two series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$

If $\sum_{k=1}^{\infty} b_k$ converges, so does $\sum_{k=1}^{\infty} a_k$, and $\sum_{k=1}^{\infty} a_k \leq \sum_{k=1}^{\infty} b_k$

If $\sum_{k=1}^{\infty} a_k$ diverges, so does $\sum_{k=1}^{\infty} b_k$

Limit Comparison Test

Suppose that $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ where c is a finite number and c>0, then either both series converge or both diverge.

Alternating Series Test

Consider the series $\sum_{k=1}^{\infty} (-1)^{k+1} c_k$ where

- $c_1 \geq c_2 \geq c_3 \geq \ldots \geq 0$;
- $\lim_{k\to\infty} c_k = 0$

Then the series converges, and its limit S lies between any two successive partial sums; that is for each $n \geq 1$, either $S_n \leq S \leq S_{n+1}$ *or* $S_{n+1} \leq S \leq S_n$. In particular $|S - S_n| < c_{n+1}$.

Absolute and Conditional Convergence

- If $\sum_{k=1}^{\infty} |a_k|$ diverges but $\sum_{k=1}^{\infty} a_k$ converges, then $\sum_{k=1}^{\infty} a_k$ *converges conditionally.*
- If $\sum_{k=1}^{\infty} |a_k|$ converges then $\sum_{k=1}^{\infty} a_k$ *converges absolutely* and $\left|\sum_{k=1}^{\infty} a_k\right| \leq \sum_{k=1}^{\infty} |a_k|$

Ratio Test

Suppose that $\sum_{k=1}^{\infty} a_k$ is a series and that $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = L$

- If L < 1, then $\sum_{k=1}^{\infty} a_k$ *converges conditionally.*
- If L > 1, (or $L \to \infty$) then $\sum_{k=1}^{\infty} a_k$ *diverges.*
- If L = 1, either convergence or divergence is possible so the test is inconclusive.

Root Test

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1,$ then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $L \to \infty$ then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii)If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1,$ then the test in inconclusive.

Power Series

Let $S(x)=\sum_{k=0}^{\infty}c_k\left(x-a\right)^k$ be a power series then there are only three possibilities:

(i) The series converges only when $x = a$.
(ii) The series converges for all x.
(iii) There is a positive number R such that the series converges if $|x - a| < R$ and diverges if $|x - a| > R$.

Derivatives and Integrals of power series.

If the power series $\sum c_k\left(x-a\right)^k$ has a radius of convergence $R > 0$, then the function f defined by

$f(x)=\sum_{k=0}^{\infty}c_k\left(x-a\right)^k$ is differentiable (and therefore continuous) on the interval $(a - R, a + R)$ and

(i) $f'(x)=c_1+2c_2\left(x-a\right)+3c_3\left(x-a\right)^2+...=\sum_{k=1}^{\infty}kc_k\left(x-a\right)^{k-1}$

(ii) $\int f(x)dx=C+c_0\left(x-a\right)+c_1\frac{\left(x-a\right)^2}{2}+c_2\frac{\left(x-a\right)^3}{3}+...=C+\sum_{k=0}^{\infty}c_k\frac{\left(x-a\right)^{k+1}}{k+1}$

The radii of convergence of the power series in Equations (i) and (ii) are both R.

Maclaurin Series

Let f be any function with infinitely many derivatives at x = 0. The **Maclaurin series** for f is the series $\sum_{k=0}^{\infty}a_k x^k$, with coefficients given by $a_k=\frac{f^{(k)}(0)}{k!}$ where $k = 0, 1, 2, \ldots$

Taylor Series

The **Taylor series** for f, expanded about $x = a$, has the form $\sum_{k=0}^{\infty}\frac{f^{(k)}\left(a\right)}{k!}\left(x-a\right)^k$.

Taylor's Theorem

If $\left|f^{(n+1)}(x)\right|\le M$ for $\left|x-a\right|\le d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality:

$$\left|R_n\left(x\right)\right|=\left|f(x)-T_n(x)\right|\le\frac{M}{(n+1)!}\left|x-a\right|^{n+1}\text{ for }\left|x-a\right|\le d$$

Familiar Limits of Sequences

1. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$

2. $\lim_{n\to\infty} \sqrt[n]{n} = 1$

3. $\lim_{n\to\infty} x^{\frac{1}{n}} = 1 \ (x > 0)$

4. $\lim_{n\to\infty} x^n = 0 \ (|x| < 1)$

5. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$

6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0$

Familiar Power Series

Function	Series	Convergence Interval	Radius of Convergence
$\sin x$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!}$	$(-\infty, \infty)$	∞
$\cos x$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-2}}{(2n-2)!}$	$(-\infty, \infty)$	∞
e^x	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$	$(-\infty, \infty)$	∞
$\frac{1}{1-x}$	$1 + x + x^2 + x^3 + x^4 + x^5 + \ldots = \sum_{n=1}^{\infty} x^{n-1}$	$(-1,1)$	1
$\frac{1}{1+x}$	$1 - x + x^2 - x^3 + x^4 - x^5 + \ldots = \sum_{n=1}^{\infty} (-1)^{n-1} x^{n-1}$	$(-1,1)$	1
$\ln(x + 1)$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$	$(-1,1]$	1
$\frac{1}{1+x^2}$	$1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \ldots = \sum_{n=1}^{\infty} (-1)^{n-1} x^{2n-2}$	$(-1,1)$	1
$\arctan x$	$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \ldots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1}$	$[-1,1]$	1