AP Calculus BC
Lesson 12.10 Taylor Series

1. (a) Find a fourth degree polynomial P such that $P(0) = P'(0) = P''(0) = P'''(0) = P^{(4)}(0) = 1$.

(b) Find an nth degree polynomial Q such that $Q^{(n)}(0) = 1$, where n is a nonnegative integer.

(c) Find an infinite polynomial R such that $R^{(n)}(0) = 1$, where n is a nonnegative integer. What is the interval of convergence for this series?

(d) What is $R'(x)$? What is the interval of convergence for this series? What is $R(x)$?

2. (a) Find a fifth degree polynomial P such that $P^{(n)}(0) = \sin^{(n)}(0)$.

(b) Extend this polynomial to a power series for $\sin(x)$. For what values of x does it converge?

(c) Find a power series for $\cos(x)$. What is the interval of convergence for this series?

3. Consider the function $g(x) = x^3 - 4x^2 + 5x + 7$.

(a) Write an equivalent polynomial of the form $g(x) = a(x - 2)^3 + b(x - 2)^2 + c(x - 2) + d$.

(b) Let f be a function with derivatives of order k for $k = 1, 2, 3, \ldots, N$ in some interval containing a as an interior point. Then, for any integer n from 0 to N, the **Taylor polynomial** of order n generated by f at a is the polynomial

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)(x-a)^k}{k!}.$$

Find the Taylor polynomial $P_3(x)$ of order (degree) 3 generated by g at 2.

(c) What connections, if any, do you see between the polynomial you found in part a) and $P_3(x)$?

4. (a) Find the Taylor polynomial of order 10 for $f(x) = \cos(x)$ at $a = 0$.

(b) Let S_k be the sum of the first k nonzero terms of the Taylor polynomial.
Draw the graphs of $y = f(x)$ and $y = P_k(x)$ for $k = 1, 2, 3, 4, 5$.

(c) Consider the power series generated by the Taylor polynomial you found. What is the interval of convergence of this power series?

5. For each function below, find the Taylor polynomials generated at the given value of a.

a. $f(x) = e^x$ at $a = 0$

b. $g(x) = \ln(1+x)$ at $a = 0$

c. $h(x) = \sin(x)$ at $a = 0$

d. $j(x) = \sin x$ at $a = \pi$

e. $k(x) = \frac{1}{x}$ at $a = 1$

6. (a) Use Taylor's Inequality (also called the Lagrange error estimation theorem) to determine how many terms of the Maclaurin series for e^x must be used to approximate e with an error no more than 10^{-6}.

Taylor's Inequality (Lagrange remainder estimation theorem)

If $\left|f^{(n+1)}(x)\right| \le M$ for $|x-a| \le d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} \qquad \text{for } |x-a| \le d$$

(b) For what values of x can we replace $\cos x$ by $1-\frac{x^2}{2!}$ with an error no more than 10^{-4}. Confirm by two methods.

7. Find the Maclaurin series for each of the following functions.

a. $f(x) = e^{-x^2}$

b. $g(x) = \sin(2x)$

c. $h(x) = \dfrac{\sin(x)}{x}$

d. $k(x) = \cos(x^2)$

8. Find the Maclaurin series for each function, and determine its radius of convergence.

a. $f(x) = (1+x)^m$ for constant m.

b. $(1+x)^5$

c. $f(x) = \sqrt{1+x}$

d. $g(x) = \left(9+x^4\right)^{-1/2}$

9. a) Use an infinite series to compute $\int_0^{0.2} e^{-x^2}\, dx$ correct to three decimal places.

b. Compute $\int_0^{1/2} f(x)\, dx$, where $f(x) = \begin{cases} \dfrac{\sin(x)}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$, accurate to three decimal places.

10. (1995BC4) Let f be a function that has derivatives of all orders for all real numbers.
Assume $f(1) = 3$, $f'(1) = -2$, $f''(1) = 2$, and $f'''(1) = 4$.

(a) Write the second-degree Taylor polynomial for f about $x = 1$ and use it to approximate $f(0.07)$.

(b) Write the third-degree Taylor polynomial for f about $x = 1$ and use it to approximate $f(1.2)$.

(c) Write the second-degree Taylor polynomial for f', the derivative of f, about $x = 1$ and use it to approximate $f'(1.2)$.

11. (1996 BC2) The Maclaurin series for $f(x)$ is given by $1+\frac{x}{2!}+\frac{x^2}{3!}+\frac{x^3}{4!}+\ \dots\ +\frac{x^n}{(n+1)!}+\ \dots$.

(a) Find $f'(0)$ and $f^{(17)}(0)$.

(b) For what values of x does the given series converge? Show your reasoning.

(c) Let $g(x) = xf(x)$. Write the Maclaurin series for $g(x)$, showing the first three nonzero terms and the general term.

(d) Write $g(x)$ in terms of a familiar function without using series. Then, write $f(x)$ in terms of the same function.

12. (1997 BC2) Let $P(x) = 7 - 3(x - 4) + 5(x - 4)^2 - 2(x - 4)^3 + 6(x - 4)^4$ be the fourth-degree Taylor polynomial for the function f about 4. Assume f has derivatives of all orders for all real numbers.

(a) Find $f(4)$ and $f'(4)$.

(b) Write the second-degree Taylor polynomial for f' around 4 and use it to approximate $f'(4.3)$.

(c) Write the fourth-degree Taylor polynomial for $g(x) = \int_4^x f(t)\, dt$ about 4.

(d) Can $f(3)$ be determined from the information given? Justify your answer.