AP Calculus BC
Lesson 4.2 The Mean Value Theorem

Remember one of the definitions we had for the derivative of a function $f(x)$ at a point $x = a$ was $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$

4.2(1)

a) Give an example of a function that is continuous at a certain point $x = a$, but does not have a derivative at $x = a$.
b) Do you think that a function could be discontinuous at a point $x = a$ and still have a derivative at $x = a$.
c) A very important theorem of Calculus is the following:

"If a function is differentiable at x = a , then the function is continuous at x = a"

Explain each step of this proof :

i) $\lim_{x\to a}\left(f(x)-f(a)\right)=\lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\cdot(x-a)\right)$

ii) $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\lim_{x\to a}(x-a)$

iii) $f'(a)\cdot(0)=0$

iv) therefore $\lim_{x\to a}\left(f(x)-f(a)\right)=0,$ so $\lim_{x\to a}f(x)=f(a)$

v) thus f is continuous at x = a.

4.2(2)

a) Consider the function f defined by $f(x)=x^2+2x-1$ on [0,1]. Show that there is a value $c\in(0,1)$ such that $f'(c)=\frac{f(1)-f(0)}{1-0}$

b) Consider the function g defined by $g(x)=\sqrt{1-\sin(x)}$ on $\left[0,\frac{\pi}{2}\right]$. Show that there is a value $c\in\left(0,\frac{\pi}{2}\right)$ such that $g'(c)=\frac{g(\pi/2)-g(0)}{\pi/2-0}$

c) Consider the function h defined by $h(x)=\frac{x^2+4x}{x-7}$ on [2,6]. Show that there is a $c\in(2,6)$ such that $h'(c)=\frac{f(6)-f(2)}{6-2}$

4.2(3) All of the examples in problem 4.11(2) above are examples of a very important calculus theorem called the **Mean Value Theorem** which states "*If f is a function that is continuous on [a,b] and differentiable on (a,b) then there exists at least one c in (a,b) such that* $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

a) Draw a picture of a non-linear continuous function on the closed interval [-2,5] and then draw a line between the points (-2,f(-2)) and (5,f(5)). Interpret the Mean Value Theorem in terms of your diagram.

b) Draw a picture of a non-linear function on the closed interval [1,4] such that f(1)=f(4) and then draw the line connecting the points (1,f(1)) and (4,f(4)). Interpret the Mean Value Theorem in terms of your diagram. Note: In this case, whenever f(a) = f(b), the Mean Value Theorem is called *Rolle's Theorem*.

c) Why is it important that the function be continuous on [a,b]? Show by a counterexample {diagram} that this condition is necessary in order for the theorem to be true.

d) Why is it important that the function be differentiable on (a,b)? Show by a counterexample {diagram} that this condition is necessary in order for the theorem to be true.

4.2(4) Find the value of c guaranteed by the mean value theorem for

$f(x) = \sqrt{1 - \sin(x)}$ on $\left[0, \frac{\pi}{2}\right]$

4.2(5) Find the value of c guaranteed by the Mean Value Theorem for

$$f(x) = \frac{x^2+4x}{x-7}$$ on [2,6]. Why doesn't the theorem apply on [2,7]?

4.2(6) When f(a) = f(b) the Mean Value Theorem is usually called Rolle's theorem.

a) Give a geometric interpretation of Rolle's theorem.
b) If f(x) = Sin(2x) on [0, 2π], find all value(s) of c guaranteed by Rolle's theorem.
c) If f(x) is a constant function on [a,b], what are the values of c, guaranteed by Rolle's theorem. What does this say about the derivative of a constant function?

4.2(7) Suppose that a car starts at time t =0 stopped and that by the end of 8 seconds the car has traveled 352 feet. Show by use of the Mean Value Theorem that at some point during the cars acceleration, the car was traveling exactly 44 feet per second.

4.2(8) A trucker is handed in a ticket at a toll booth showing that in 2 hours the truck had covered 159 miles on a toll road on which the speed limit was 65 mph. Show that at sometime during the trip the trucker was speeding.