AP Calculus BC
Lesson 5.2 The Signed Area Function

If f is a function defined for all $a \le x \le b$,
then $\int_a^b f(x)dx$ denotes the **signed** area bounded by $x = a$, $x = b$, $y = f(x)$, and the x-axis.

1. Use geometry to find each of the following:

(a) $\int_1^5 (2x+1)\,dx$

(b) $\int_2^5 |x-3|\,dx$

(c) $\int_{-3}^3 \sqrt{9-x^2}\,dx$

(d) $\int_2^5 6dx$

(e) $\int_0^{2\pi} \sin(x)\,dx$

2. Given that $\int_a^b f(x)dx = 4$, $\int_a^c f(x)dx = -2$, and $\int_b^d f(x)dx = 7$, find each of the following:

(a) $\int_b^c f(x)dx$

(b) $\int_a^d f(x)dx$

(c) $\int_c^d f(x)dx$

3. Use the fact that $\int_0^{\pi} \sin(x)\,dx = 2$ to calculate each of the following:

(a) $\int_0^{\pi/2} \sin(x)\,dx$

(b) $\int_0^{\pi} 2\sin(x)\,dx$

(c) $\int_0^{\pi/2} \sin(2x)\,dx$

(d) $\int_{\pi}^{2\pi} 3\sin(x)\,dx$

(e) $\int_0^{\pi} \left|\sin(2x)\right|\,dx$

(f) $\int_0^{\pi} (\sin(x)+3)\,dx$

4. Find the average value of the function on the given interval.

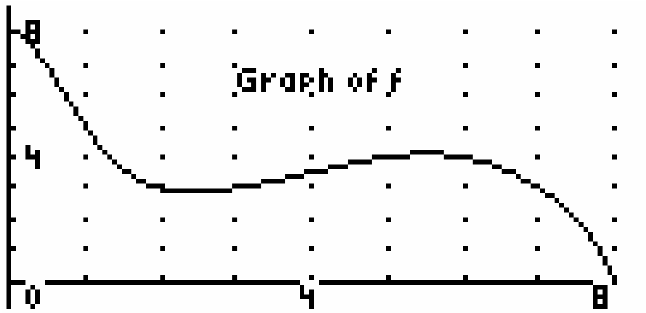
(a) $f(x) = 2x+1$ on [1,4]

(b) $f(x) = |x-2|$ on [0,5]

(c) $f(x) = \sin(x)$ on [0,2π]

(d) $f(x) = \sqrt{4-x^2}$ on [-2,0]

5. The graph of a function *f* is shown below:



(a) Which of the following is the best estimate for $\int_1^6 f(x)dx$: -24, 9, 20, or 38? Why?

(b) Find positive integers *A* and *B* such that $A \le \int_3^7 f(x)dx \le B$. Explain!

(c) $\int_6^8 f(x)dx \approx 4$. Does this approximation overestimate or underestimate the exact value? Explain.

(d) Estimate the average value of *f* over the interval [0,2].

6. For each integral listed, estimate the value of the integral by using four subintervals of equal width using the right, left, and midpoint approximations. Use your calculator to find the values of each approximation.

a) $\int_0^2 \left(x^2 - 1\right) dx$

b) $\int_0^1 -x^2 dx$

c) $\int_{-\pi}^{\pi} (\sin x + 1) dx$

7. Write each of the following limits as a definite integral:

a) $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{4}{n}\left(\frac{4k}{n}\right)^2$ b) $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{2}{n}\ln\left(3+\frac{2k}{n}\right)$ c) $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\left(\frac{k}{n}-2\right)^3$

d) $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{8k^2}{n^3}$ e) $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n+k}{n^2}$

8. Write each integral as the limit of a Riemann sum:

a) $\int_1^3 \sin^2 x\, dx$ b) $\int_2^5 (x^2+3x-1)dx$

9. Find the exact value of each integral by using geometry. Use your calculator to check your solution.

a) $\int_{-1}^{1}\sqrt{1\text{-}x^2}\, dx$ b) $\int_0^2 \sqrt{4\text{-}x^2}\, dx$

c) $\int_{-1}^{1}(1-|x|)dx$ d $\int_{-1}^{1}(1+\sqrt{1-x^2})dx$

10. Sketch a graph of the area indicated by each integral, and use area to evaluate the integral.

a) $\int_{-2}^{3}\frac{|x|}{x}dx$ b) $\int_{-3}^{4}\frac{x^2-1}{x+1}dx$

11. Let $S_n=\frac{1^2}{n^3}+\frac{2^2}{n^3}+\frac{3^2}{n^3}+\frac{4^2}{n^3}+\cdots+\frac{(n-1)^2}{n^3}+\frac{n^2}{n^3}$.

Interpret S_n as a Riemann sum, and write $\lim_{n\to\infty}S_n$ as a definite integral.

12. Find a closed form for $A_f(x)$, and evaluate each area function at $x = 5$ and $x = -3$.

(a) $\int_0^x 4dt$

(b) $\int_3^x 4dt$

(c) $\int_0^x 3tdt$

(d) $\int_3^x 3tdt$

(e) $\int_0^x (3-t)dt$

13. Let $A_f(x) = \int_0^x (6-t^2)\,dt$. Use the graph of $f(t) = 6-t^2$ to answer the following:

(a) On what interval(s) is $A_f(x)$ increasing?

(b) At what value(s) of x does $A_f(x)$ have a maximum? a minimum? How do you know?

(c) On what interval(s) is $A_f(x)$ concave up? How do you know?

(d) Draw the graph of $A_f(x) = \int_0^x (6-t^2)\,dt$. (Hint: The syntax is $\int(6-t^2, t, 0, x)$.) What function is this? Why?

(e) Consider $g(x) = \int_3^x (6-t^2)\,dt$. How would the graph of $g(x)$ differ from that of $A_f(x)$? Draw the graph of $y = g(x)$ to test your conjecture.