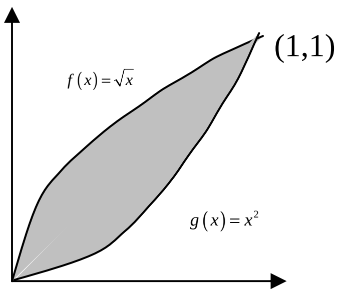
BC Calculus
A summary of calculating volumes of revolution

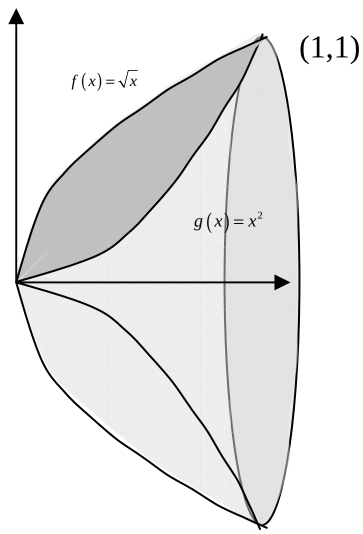
Example:

Find the volume of the solid obtained by rotating the region bounded by the graphs of $f(x) = \sqrt{x}$ and $g(x) = x^2$ about the x-axis.

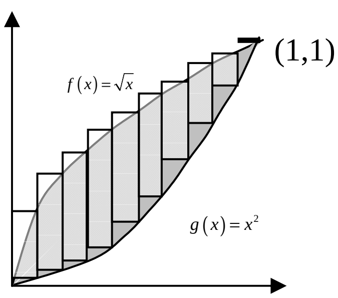


The region

Rotated about the x-axis

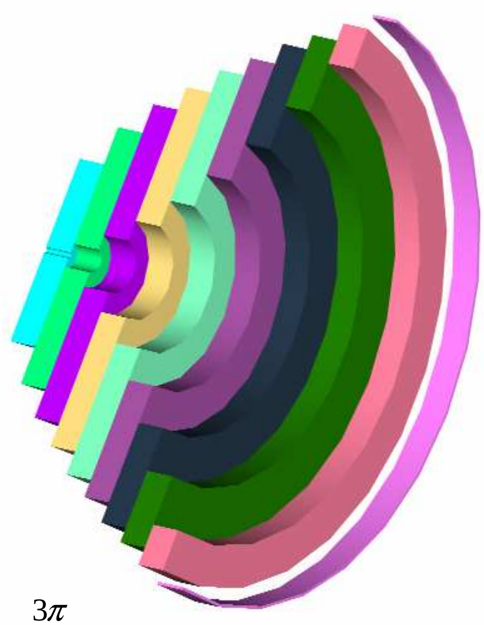


Washers



Right-hand rectangles

Cutaway isometric view



The volume of each washer is:

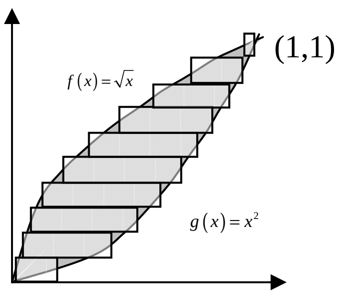
$$\pi R^2 - \pi r^2 = \pi\left(\left(\sqrt{x_i}\right)^2 - \left(x_i^{\,2}\right)^2\right) \bullet \Delta x_i$$

So the Riemann sum is $\pi \sum_{i=1}^{\infty}\left(\left(\sqrt{x_i}\right)^2 - \left(x_i^{\,2}\right)^2\right) \bullet \Delta x_i$

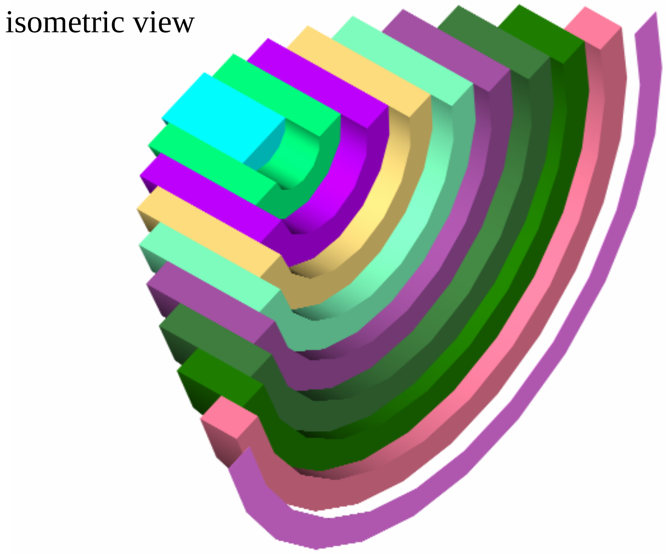
Which yields the integral: $V = \pi \int_0^1 \left(\left(\sqrt{x}\right)^2 - \left(x^2\right)^2\right) dx = \dfrac{3\pi}{10}$

Shells

Cutaway isometric view



"Midpoint" rectangles



The volume of each shell is:

$$\pi R^2 h - \pi r^2 h = \pi {y_{i+1}}^2 \left(g^{-1}(y_m) - f^{-1}(y_m)\right) - \pi y_i^2 \left(g^{-1}(y_m) - f^{-1}(y_m)\right)$$

So the sum becomes: $\pi \sum_{i=0}^{\infty} {y_{i+1}}^2 \left(g^{-1}(y_m) - f^{-1}(y_m)\right) - y_i^2 \left(g^{-1}(y_m) - f^{-1}(y_m)\right)$

Simplifying this expression gives the Riemann sum:

$$\pi \sum_{i=0}^{\infty} \left(g^{-1}(y_m) - f^{-1}(y_m)\right)\left({y_{i+1}}^2 - y_i^2\right) = \pi \sum_{i=0}^{\infty} \left(g^{-1}(y_m) - f^{-1}(y_m)\right)(y_{i+1} - y_i)(y_{i+1} + y_i)$$

$$= \pi \sum_{i=0}^{\infty} \left(g^{-1}(y_m) - f^{-1}(y_m)\right) 2 \bullet (y_{i+1} - y_i)\left(\frac{y_{i+1} + y_i}{2}\right) = \pi \sum_{i=0}^{\infty} \left(g^{-1}(y_m) - f^{-1}(y_m)\right) 2 \bullet \Delta y_i \bullet y_m$$

$$= 2 \bullet \pi \sum_{i=0}^{\infty} \left(g^{-1}(y_m) - f^{-1}(y_m)\right)_i \bullet y_m \bullet \Delta y$$

which, in this case is the integral: $V = 2\pi \int_0^1 y\left(\sqrt{y} - y^2\right) dy = \frac{3\pi}{10}$

Note: An easier way to think of this is to imagine each shell cut open and unfolded into a solid that is approximated by a rectangular prism. The volume of each prism would then be $V = lwh = C \bullet f(y) \bullet dy = 2\pi y \bullet f(y) \bullet dy$ so the sum is $V = 2\pi \int_0^1 y\left(\sqrt{y} - y^2\right) dy = \frac{3\pi}{10}$.