AP Calculus BC
Lesson 7.7 L'Hopital's Rule

L'Hopital's Rule will help us to evaluate analytically certain limits of the following indeterminate forms:

a) $\lim_{x\to a}\frac{f(x)}{g(x)}$ where $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=0$.

b) $\lim_{x\to a}\frac{f(x)}{g(x)}$ where $\lim_{x\to a} f(x)=\infty$ and $\lim_{x\to a} g(x)=\infty$.

To see how L'Hopital's Rule works do the following:

1. Given the limit $\lim_{x\to 0}\frac{\sin(5x)}{2x}$.

   a) Show that this is one of the forms mentioned above.

   b) Graph each of the following, where $f(x)=\sin(5x)$ and $g(x) = 2x$ using $x \in$ [-1,1] and $y \in$ [-1,4].

   (i) $\frac{f(x)}{g(x)}$ (ii) $\frac{f'(x)}{g'(x)}$

   c) Even though (i) and (ii) are very different graphs, what do you notice about them that is the same? Explain.

   d) Find $\lim_{x\to 0}\frac{f'(x)}{g'(x)}$ analytically.

Problem 1 contains the essence of L'Hopital's rule which states that if you are trying to find the limit of one of the indeterminate forms mentioned above, you can try to find instead the limit $\lim_{x\to a}\frac{f'(x)}{g'(x)}$. Its value will be the same as that of the indeterminate form. This second limit may often be calculated analytically.

2. Check to see that each problem is an indeterminant form and if so, then evaluate by using L'Hopital's rule.

1. $\lim_{x\to 2} \dfrac{x-2}{x^2-4}$

2. $\lim_{x\to 1} \dfrac{x^3-1}{4x^3-x-3}$

3. $\lim_{x\to 0} \dfrac{\sin(5x)}{4x}$

4. $\lim_{x\to \infty} \dfrac{3x^2-1}{2x^2-x-4}$

5. $\lim_{x\to \pi/2} \dfrac{2x-\pi}{\cos(x)}$

6. $\lim_{x\to 0} \dfrac{\cos(x)-1}{x^2}$

7. $\lim_{x\to 0^+} \dfrac{2x}{x+7\sqrt{x}}$

8. $\lim_{x\to 0} \dfrac{x(1-\cos(x))}{x-\sin(x)}$

9. $\lim_{x\to\infty} \frac{(\ln(x))^3}{x}$

10. $\lim_{x\to 2}(2x-1)\tan(\pi x)$

11. $\lim_{x\to 1}\left(\frac{1}{\ln(x)}-\frac{1}{x-1}\right)$

12. $\lim_{x\to 0}\left(\frac{1}{\sin(x)}-\frac{1}{x}\right)$

13. $\lim_{x\to 0^+}\left(1+x\right)^{1/x}$

14. $\lim_{x\to 0}\left(e^x+x\right)^{1/x}$