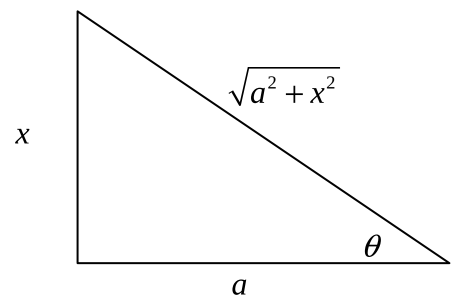
AP Calculus BC
Lesson 8.3
Trigonometric Substitutions

Today we consider a way to substitute trigonometric functions in order to help us integrate functions containing terms of the form $\left(a^2+x^2\right)^n$, $\left(a^2-x^2\right)^n$, and $\left(x^2-a^2\right)^n$, where *a* is a constant and *n* is a rational exponent.

Case I: Integrals containing $\left(a^2+x^2\right)^n$:

We substitute for each *x*, $x = a\tan(\theta)$, which would make $dx = a\sec^2(\theta)d\theta$.
We will also use the following reference triangle.



Example: $\displaystyle\int \frac{dx}{\sqrt{4+x^2}}$

We substitute $x = 2\tan(\theta)$ and $dx = 2\sec^2(\theta)d\theta$ to obtain

$$\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{2\sec^2(\theta)d\theta}{\sqrt{4+4\tan^2(\theta)}} = \int \frac{2\sec^2(\theta)d\theta}{2\sec(\theta)} = \int \sec(\theta)d\theta$$
$$= \ln\left|\sec(\theta)+\tan(\theta)\right| + C$$

(The last equality is valid because the derivative of the right hand side is sec(θ).)
We now use our reference triangle to get back to a function of *x* instead of θ.

$$\int \frac{dx}{\sqrt{4+x^2}} = \ln\left|\frac{\sqrt{4+x^2}}{2}+\frac{x}{2}\right| + C$$

Compare this to the TI-89 answer of $\ln\left|\sqrt{x^2+4}+x\right|$

Note: $\ln\left|\frac{\sqrt{4+x^2}}{2}+\frac{x}{2}\right| = \ln\left|\frac{1}{2}\left(\sqrt{4+x^2}+x\right)\right| = \ln\frac{1}{2} + \ln\left|\left(\sqrt{4+x^2}+x\right)\right|$

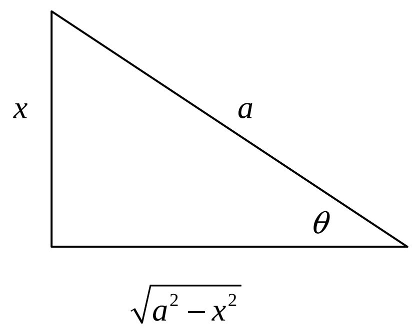
Note: It would be helpful if you remember the following.

(1) $\int \sec(\theta)d\theta = \ln\left|\sec(\theta)+\tan(\theta)\right| + C$

(2) $\int \csc(\theta)d\theta = \ln\left|\csc(\theta)-\cot(\theta)\right| + C$

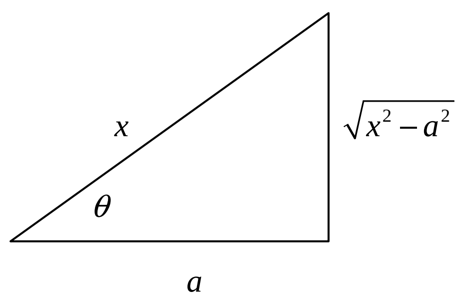
Case II: Integrals containing $(a^2 - x^2)^n$**:**

We substitute for each x, $x = a\sin(\theta)$, which would make $dx = a\cos(\theta)d\theta$.
We will also use the following reference triangle.



Case III: Integrals containing $(x^2 - a^2)^n$**:**

We substitute for each x, $x = a\sec(\theta)$, which would make $dx = a\sec(\theta)\tan(\theta)d\theta$. We will also use the following reference triangle.



8.3(1) Use a trigonometric substitution to integrate each of the following:
Check with technology.

1. $\int \frac{x^2 dx}{\sqrt{9-x^2}}$

2. $\int \frac{dx}{4+x^2}$

3. $\int \sqrt{25-x^2}\,dx$

4. $\int \frac{dx}{\sqrt{1-4x^2}}$

5. $\int \frac{3dx}{\sqrt{9x^2-1}}$

6. $\int \frac{dx}{\sqrt{x^2-25}}$

Check carefully against the TI-89

7. $\int \frac{4x^2 dx}{(1-x^2)^{3/2}}$

you ARE smarter than your calculator! ☺

8.3(2) Sometimes before making a trigonometric substitution it pays to complete the square and then make a traditional *u*-substitution.

1. $\int \frac{dx}{\sqrt{x^2 - 2x}}$

2. $\int \frac{(x-1)dx}{\sqrt{2x - x^2}}$

3. $\int_1^4 \frac{dy}{y^2 - 2y + 10}$

A final note: Two of these integral types occur so often that it is a good idea to know them. These are the following:

$$\int \frac{du}{u^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$$