Openers for AP Calculus BC
Lesson 8.8 Improper Integrals

Improper Integrals: A definite integral is said to be improper if any of the following occur:

1. The interval over which you are integrating is infinite.

2. The integrand, i.e., the function that you are integrating, is unbounded on the interval over which you are integrating.

Example 1: $\int_1^\infty \frac{dx}{x^2} = \lim_{b\to\infty}\int_1^b \frac{dx}{x^2} = \lim_{b\to\infty}\frac{-1}{x}\Big|_1^b = \lim_{b\to\infty}\left(1-\frac{1}{b}\right)=1$

Because the limit exists, we say that this improper integral ***converges***.

Example 2: $\int_1^\infty \frac{dx}{x} = \lim_{b\to\infty}\int_1^b \frac{dx}{x} = \lim_{b\to\infty}\ln(x)\Big|_1^b = \lim_{b\to\infty}\ln(b)=\infty$

Because the limit does not exist, we say that this improper integral ***diverges.***

8.8(1) Use limits to determine whether each of the following improper integrals converge or diverge. (You must use a limit analysis for each of these)

1. $\int_0^\infty \frac{dx}{x^2+4}$

2. $\int_0^1 \frac{dx}{\sqrt{x}}$

3. $\int_{-1}^1 \frac{dx}{x^{2/3}}$

4. $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

5. $\int_2^\infty \frac{2dx}{x^2 - x}$

6. $\int_0^{\frac{\pi}{2}} \tan(x)dx$

7. $\int_\pi^\infty \frac{2+\cos(x)}{x}dx$

8.8(2) {1985 BC #5} Let f be the function defined by $f(x) = -\ln(x)$ for $0 < x \leq 1$ and let R be the region between the graph of f and the x-axis.

a. Determine whether the region R has finite area. Justify your answer.

b. Determine whether the solid generated by revolving region R about the y-axis has finite volume. Justify your answer.

8.8(3) How do we deal with problems such as $\int_1^\infty \frac{2+\cos(x)}{x^2}dx$?

The first thing is to determine whether or not it converges.

a) Draw the graphs of $f(x) = \frac{2+\cos(x)}{x^2}$ and $g(x) = \frac{3}{x^2}$ with x:[1,20] y:[0,0.5] What do you notice about the two graphs. What do the two graphs tell you about the relative size of $\int_1^\infty \frac{2+\cos(x)}{x^2}dx$ and $\int_1^\infty \frac{3}{x^2}dx$?

b) Evaluate $\int_1^\infty \frac{3}{x^2}dx$. What do you now know about $\int_1^\infty \frac{2+\cos(x)}{x^2}dx$?

c) From your results in part b) we know it is sensible to use technology to find an approximation to the improper integral $\int_1^\infty \frac{2+\cos(x)}{x^2}dx$. We can divide the integral

$$\int_1^\infty \frac{2+\cos(x)}{x^2}dx = \int_1^{20} \frac{2+\cos(x)}{x^2}dx + \int_{20}^\infty \frac{2+\cos(x)}{x^2}dx .$$

Any number could be chosen to break the integral up; we just chose 20 because we used 20 for graphing.

The first integral is proper and may be evaluated. The second integral is still improper and is called the **tail** integral. Approximate the proper integral using technology.

d) How much error do we have by leaving off the tail integral?
Hint: Find an upper bound for the tail integral using something that you know is larger than the tail integral.

8.8(4) Determine whether each of the following integrals converges or diverges. Do not evaluate.

a) $\int_1^\infty \frac{dx}{e^x + 1}$

b) $\int_2^\infty \frac{dx}{\ln(x)}$

c) $\int_1^\infty \frac{dx}{x^3 + 1}$

d) $\int_0^\infty e^{-x} \cos(x) dx$

e) $\int_2^\infty \frac{dx}{\sqrt{x-1}}$

8.8(5) For each integral you found to be convergent in the previous problem do the following:

a) Find an approximation to the improper integral using a dividing point of 10 each time.
b) Determine an upper bound on the error made by leaving off the **tail** integral.
c) Use technology to evaluate the original integral expression. *In some cases an exact answer is possible.*

8.8(6) Find a smallest integer n such that the error in computing $\int_1^\infty \frac{dx}{e^x + 1}$ by using your calculator and then leaving off the tail integral will be less than .001.

Note: Here we assume that using the approximate answer given by your calculator on the proper integral gives us essentially no error and that all the error is from leaving off the tail integral.