AP Calculus BC
Review — Chapter 11 (Parametric Equations and Polar Coordinates)

Things to Know and Be Able to Do

- Understand the meaning of equations given in parametric and polar forms, and develop a sketch of the appropriate graph
- Find the slope of a tangent line to a parametrically-given curve at a particular point using the equation $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}$, and understand how this is derived using the Chain Rule
- Find the second derivative of a parametrically-given curve at a particular point using the equation $\frac{d^2y}{dx^2}=\frac{\frac{d}{dt}\left(dy/dx\right)}{dx/dt}$, and understand how this is derived using the Chain Rule
- Find the area enclosed by a parametrically-given curve
- Find the arc length of a parametrically-given curve and the area of the surface generated by revolving a parametrically-given curve
- Find the area enclosed by a polar curve using the equation $A=\frac{1}{2}\int r^2 d\theta$ (understanding that $r=f(\theta)$) and understand how this is derived. Be very careful with finding the endpoints for evaluating the integral!
- Find the arc length of a polar curve

Practice Problems

These problems should be done without a calculator, with the exception of **12**, *as explained below. The original test, of course, required that you show relevant work for free-response problems.*

1 For $0\le t\le 13$, an object travels along an elliptical path given parametrically by $\begin{cases} x=3\cos t \\ y=4\sin t \end{cases}$. At the point at which $t=13$, the object leaves the path and travels along the line tangent to the path at that point. What is the slope of the line on which the object travels?

a $-\frac{4}{3}$ **b** $-\frac{3}{4}$ **c** $-\frac{4\tan 13}{3}$ **d** $-\frac{4}{3\tan 13}$ **e** $-\frac{3}{4\tan 13}$

2 The position of a particle moving in the *xy*-plane is given by the parametric equations $\begin{cases} x=t^3-3t^2 \\ y=2t^3-3t^2-12t \end{cases}$. For what values of *t* is the particle at rest?

a -1 only **b** 0 only **c** 2 only **d** -1 and 2 only **e** -1, 0, and 2

3 A curve *C* is defined by the parametric equations $\begin{cases} x=t^2-4t+1 \\ y=t^3 \end{cases}$. Which of the following is an equation of the line tangent to the graph of *C* at the point $(-3,8)$?

a $x=-3$ **b** $x=2$ **c** $y=8$ **d** $y=-\frac{27}{10}(x+3)+8$ **e** $y=12(x+3)+8$

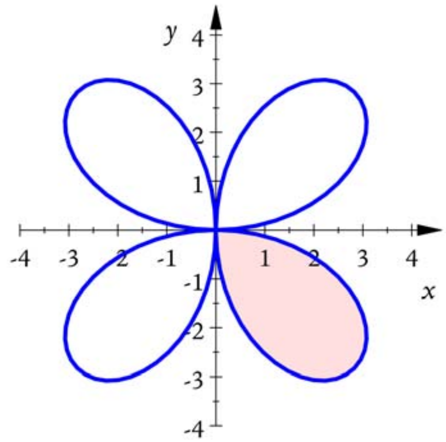
4 A particle moves so that its position at time t is given by $\begin{cases} x = t^2 \\ y = \sin(4t) \end{cases}$. What is the speed of the particle when $t = 3$?

a $-8\sin 12$ **b** $\dfrac{4\cos 12}{6}$ **c** $\sqrt{(4\cos 12)^2 + 36}$ **d** $\sqrt{(\sin 12)^2 + 81}$ **e** $(4\cos 12)^2 + 36$

5 Which of the following integrals represents the area shaded in the graph shown at right? The curve is given by $r = 4\sin 2\theta$.

a $\int_{3\pi/2}^{2\pi} 2\sin(2\theta)\,d\theta$ **b** $\int_{\pi/2}^{\pi} 8\sin^2(2\theta)\,d\theta$ **c** $\int_0^{\pi} 2\sin^2(2\theta)\,d\theta$

d $\int_{\pi/2}^{\pi} 2\sin(2\theta)\,d\theta$ **e** $\int_{3\pi/2}^{2\pi} 4\sin^2(2\theta)\,d\theta$



6 Which of the following integrals represents the arc length of the polar function $r = 1 + \cos\theta$ from $0 \le \theta \le \pi$?

a $\int_0^{\pi} \sqrt{(1+\cos\theta)^2 + (-\sin\theta)^2}\,d\theta$ **b** $\int_0^{\pi} \sqrt{1+\sin^2\theta}\,d\theta$

c $\int_0^{\pi} (1+\cos\theta)\,d\theta$ **d** $\int_0^{\pi} \tfrac{1}{2}(1+\cos\theta)^2\,d\theta$

e $\int_0^{\pi} 2\pi(1+\cos\theta)\sin\theta\sqrt{(1+\cos\theta)^2 + (-\sin\theta)^2}\,d\theta$

7 Consider the graph of the vector function $\mathbf{r}(t) = \langle 1+t^3, 3+4t \rangle$. What is the value of $\dfrac{d^2y}{dx^2}$ at the point on the graph where $x = 2$?

a 0 **b** $\dfrac{4}{3}$ **c** $-\dfrac{8}{3}$ **d** $-\dfrac{8}{9}$ **e** $-\dfrac{1}{18}$

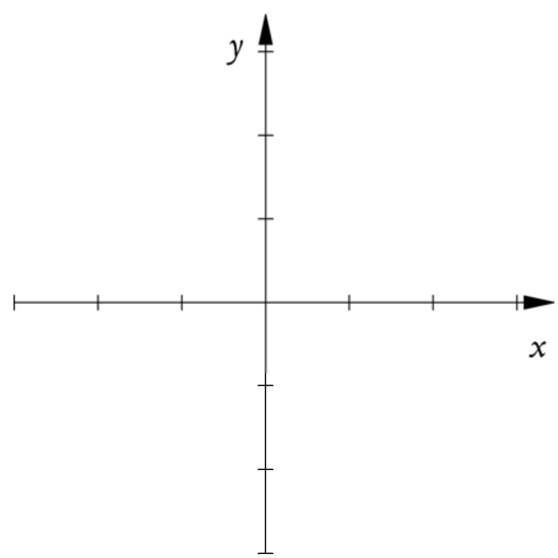
8 A particle moves so that at time $t > 0$ its position vector is $\langle \ln(t^2+2t), 2t^2 \rangle$. At time $t = 2$, its velocity vector is

a $\langle \tfrac{3}{4}, 8 \rangle$ **b** $\langle \tfrac{3}{4}, 4 \rangle$ **c** $\langle \tfrac{1}{8}, 8 \rangle$ **d** $\langle \tfrac{1}{8}, 4 \rangle$ **e** $\langle -\tfrac{5}{16}, 4 \rangle$

9 Consider the curves $r_1 = 2\cos\theta$ and $r_2 = \sqrt{3}$.

a Sketch the curves on the axes provided at right.

b Show use of calculus to find the area of the region common to both graphs.



10 Consider the curve given parametrically by $\begin{cases} x = 2t^3 - 3t^2 \\ y = t^3 - 12t \end{cases}$.

a In terms of t, find $\dfrac{dy}{dx}$.

b Write an equation for the line tangent to the curve at the point at which $t = -1$.

c Find the x- and y-coordinates for each critical point on the curve and identify each point as having a vertical or horizontal tangent.

11 A projectile is launched from the edge of a cliff 192 feet above the ground below. The projectile has an initial velocity of 128 feet per second and is launched at an angle of 30° to the horizontal. A horizontal wind is also blowing against the projectile at a velocity of 18 feet per second. The acceleration of gravity is 32 feet per second per second.

a Sketch the situation and the path of the projectile.

b Write a vector equation that represents the path of the project tile, using t in seconds as the parameter.

c When does the projectile reach its maximum height, and what is this maximum height? Justify your answer.

d How far, horizontally, does the projectile travel before landing on the ground?

When this test was originally administered, the following question was to be taken home and completed with the use of a calculator but with no other resources permitted to be used, such as other people, books, computers, or anything else. Students had one weekend to complete this.

12 An object moving along a curve in the xy-plane has position $(x(t), y(t))$ at time $t \geq 0$ with $\frac{dx}{dt} = 12t - 3t^2$ and $\frac{dy}{dt} = \ln\left(1 + (t-4)^4\right)$. At time $t = 0$, the object is at position $(-13, 5)$. At time $t = 2$, the object is at point P with x-coordinate 3.

a Find the acceleration vector and the speed at time $t = 2$.

b Find the y-coordinate of point P.

c Write an equation for the line tangent to the curve at point P.

d For what value(s) of t, if any, is the object at rest? Justify your answer.

Answers

1 d	**2** c	**3** a	**4** c
5 b	**6** a	**7** d	**8** a

9a see solutions **9b** $\frac{7\pi}{6}-\frac{\sqrt{3}}{2}$

10a $\frac{3t^2-12}{6t^2-6t}=\frac{t^2-4}{2t^2-2t}$ **10b** $y-11=-\frac{3}{4}(x+5)$

10c horizontal at $(-28,16)$ and $(4,-16)$;
vertical at $(-1,-11)$ and $(0,0)$

11a see solutions

11b $\mathbf{r}(t)=\left\langle\left(64\sqrt{3}-18\right)t,64t-16t^2+192\right\rangle$

11c 256 ft at $t=2$ s **11d** $384\sqrt{3}-108$ ft

12a $\mathbf{a}(2)=\left\langle 0,-\frac{32}{17}\right\rangle$; $\|\mathbf{v}(2)\|=\sqrt{144+(\ln 17)^2}\approx 12.330$

12b 13.671 **12c** $y-13.671=\frac{\ln 17}{12}(x-3)$

12d $t=4$ only

Solutions

1 We find $\frac{dx}{dt}=-3\sin t$ and $\frac{dy}{dt}=4\cos t$. Then (by the Chain Rule) $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{4\cos t}{-3\sin t}=-\frac{4}{3\tan t}$, so at $t=13$, $\frac{dy}{dx}=-\frac{4}{3\tan 13}$. This is choice **d**.

2 We want a time t at which both $\frac{dx}{dt}=0$ and $\frac{dy}{dt}=0$; that is, both $3t^2-6t=0$ and $6t^2-6t-12=0$. The first equation is $3t^2=6t$ or $t=2$, and the second equation is the same as $t^2-t-2=0$, which factors to $(t-2)(t+1)=0$. Therefore $\frac{dy}{dt}=0$ for $t\in\{2,-1\}$; the intersection of the two solution sets is $t=2$ only, choice **c**.

3 Firstly, find that $(x,y)=(-3,8)$ occurs at $t=2$. Now find $\frac{dx}{dt}=2t-4$ and $\frac{dy}{dt}=3t^2$, so $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{3t^2}{2t-4}$. Then $\left.\frac{dy}{dx}\right|_{t=2}=\frac{3(2)^2}{2(2)-4}=\frac{12}{0}$. This is undefined, so the tangent line must be vertical. Since the point has an x-value of -3, the relevant tangent line is $x=-3$, choice **a**.

4 The particle's velocity is given by $\mathbf{v}(t)=\left\langle\frac{dx}{dt},\frac{dy}{dt}\right\rangle=\langle 2t,4\cos 4t\rangle$. We are interested in $\|\mathbf{v}(3)\|=\|\langle 6,4\cos(12)\rangle\|$, which is $\sqrt{6^2+(4\cos 12)^2}=\sqrt{(4\cos 12)^2+36}$, choice **c**.

5 Recall that the area enclosed by a polar graph is given by $\frac{1}{2}\int r^2\,d\theta$. Therefore the integrand must be $\frac{1}{2}(4\sin 2\theta)^2=\frac{1}{2}(16)\sin^2 2\theta=8\sin^2 2\theta$. Without even having to consider the limits, we already know that the correct choice must be **b**.

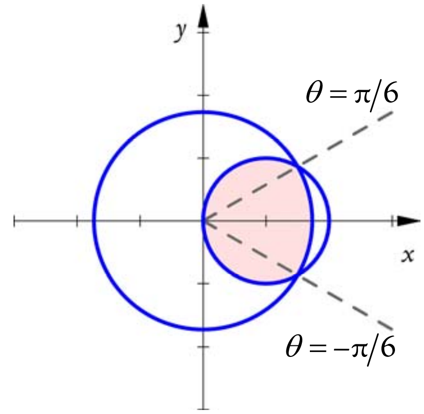
6 Recall that the arc length of a curve represented in polar coordinates is given by $\int\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$. Since $\frac{dr}{d\theta}=-\sin\theta$, the integral must take the form $\int\sqrt{(1+\cos\theta)^2+(-\sin\theta)^2}\,d\theta$. We are given the limits, and the correct answer is **a**.

7 Recall that when a curve is given parametrically, $\dfrac{d^2y}{dx^2}=\dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$. Therefore we must find $\dfrac{dy}{dx}=\dfrac{dy/dt}{dx/dt}$. This is $\dfrac{4}{3t^2}$.

Therefore $\dfrac{d^2y}{dx^2}=\dfrac{\frac{d}{dt}\left(\frac{4}{3t^2}\right)}{3t^2}=\dfrac{-\frac{8}{3t^3}}{3t^2}=-\dfrac{8}{9t^5}$. The point at which $x=2$ is the point at which $t=1$, so $\left.\dfrac{d^2y}{dx^2}\right|_{t=1}=-\dfrac{8}{9}$, choice **d**.

8 The velocity vector is $\mathbf{v}(t)=\left\langle \dfrac{dx}{dt},\dfrac{dy}{dt}\right\rangle=\left\langle \dfrac{2t+2}{t^2+2t},4t\right\rangle$. At $t=2$, we have $\mathbf{v}(2)=\left\langle \dfrac{3}{4},8\right\rangle$, choice **a**.

9a The first equation is a circle of diameter 2 tangent to the *y*-axis and shifted along the positive *x*-axis; the second is a circle of diameter $\sqrt{3}$ centered at the origin. The circles along with the region referred to in part **b** are shown at right.



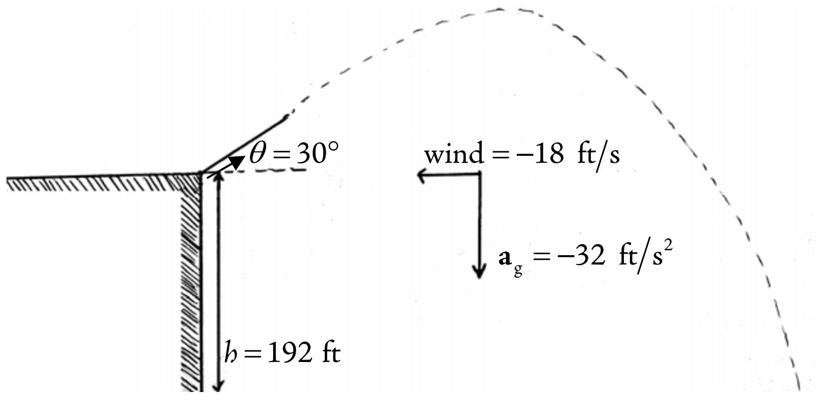
9b As shown in the diagram, the circles intersect at the points $(r,\theta)=\left(\sqrt{3},\dfrac{\pi}{6}\right)$ and $\left(\sqrt{3},-\dfrac{\pi}{6}\right)$. Note that the region is symmetric about the line $\theta=0$, so we have the luxury of considering only the portion of it with $\theta\in\left[0,\dfrac{\pi}{2}\right)$ and then doubling its area. For $\theta\in\left[0,\dfrac{\pi}{6}\right)$, the outer limit of the region is $r_2=\sqrt{3}$, so that part's area is $\int_0^{\pi/6}\frac{1}{2}\left(\sqrt{3}\right)^2 d\theta=\dfrac{\pi}{4}$. For $\theta\in\left[\dfrac{\pi}{6},\dfrac{\pi}{2}\right)$, the outer limit of the region is $r_1=2\cos\theta$, so that part's area is $\int_{\pi/6}^{\pi/2}\frac{1}{2}(2\cos\theta)^2\,d\theta=\int_{\pi/6}^{\pi/2}2\cos^2\theta\,d\theta$. To evaluate this, recall the identity $\cos^2\theta=\frac{1}{2}+\frac{1}{2}\cos 2\theta$. Therefore the integral may also be written as $\int_{\pi/6}^{\pi/2}(1+\cos 2\theta)\,d\theta \;=\theta+\frac{1}{2}\sin 2\theta\Big]_{\pi/6}^{\pi/2}=\dfrac{\pi}{2}-\left(\dfrac{\sqrt{3}}{4}+\dfrac{\pi}{6}\right)=\dfrac{\pi}{3}-\dfrac{\sqrt{3}}{4}$. The union of these two regions is then $\dfrac{\pi}{4}+\dfrac{\pi}{3}-\dfrac{\sqrt{3}}{4}=\dfrac{7\pi}{12}-\dfrac{\sqrt{3}}{4}$, which is half of the area of the entire region. Therefore the area of the whole region is $\dfrac{7\pi}{6}-\dfrac{\sqrt{3}}{2}$.

10a The Chain Rule gives $\dfrac{dy}{dx}=\dfrac{dy/dt}{dx/dt}$. This is $\dfrac{3t^2-12}{6t^2-6t}$. If you are so inclined, you may factor out a 3 from the numerator and a 6 from the denominator, yielding $\dfrac{t^2-4}{2t^2-2t}$, but such simplification is entirely optional.

10b We need $\left.\dfrac{dy}{dx}\right|_{t=-1}=\dfrac{3(-1)^2-12}{6(-1)^2-6(-1)}=\dfrac{-9}{12}=-\dfrac{3}{4}$. We also must find *x* and *y* values: $x(-1)=2(-1)^3-3(-1)^2$ $=-5$ and $y(-1)=(-1)^3-12(-1)=11$. Therefore an equation for the tangent line is given by, in point-slope form, $y-11=-\dfrac{3}{4}(x+5)$.

10c Critical points are those at which the curve's derivative is either undefined (vertical tangent line) or zero (horizontal tangent line). The derivative will be undefined iff (if and only if) its denominator is zero; that is, $\dfrac{dx}{dt}=0$, so we solve $2t^2-2t=0$; this gives $t\in\{0,1\}$. At $t=0$, $(x,y)=(0,0)$; at $t=1$, $(x,y)=(-1,-11)$. Therefore the curve has vertical tangent lines at $(-1,-11)$ and $(0,0)$. The derivative is zero iff its numerator is zero; that is, $\frac{dy}{dt}=0$, so we solve $3t^2-12=0$ to get $t=\pm 2$. At $t=-2$, $(x,y)=(-28,16)$, and at $t=2$, $(x,y)=(4,-16)$. Therefore the curve has horizontal tangent lines at $(-28,16)$ and $(4,-16)$.

11a An annotated hand-sketch is presented below. It is very much not to scale, and the path should be symmetric and parabolic.



11b The component of the initial velocity in the horizontal direction is $128\cos 30°$, and the wind slows the projectile down by 18 feet every second. Therefore $x(t)=(128\cos 30°)t-18t=\left(64\sqrt{3}-18\right)t$. In the vertical direction, the initial velocity's component is $128\sin 30°$, the particle has an initial height of 192 ft, and gravity's contribution is given by $\frac{1}{2}a_g t^2=\frac{1}{2}(32)t^2=16t^2$. The total is $y(t)=(128\sin 30°)t-16t^2+192=64t-16t^2+192$. Therefore the vector equation is $\mathbf{r}(t)=\left\langle\left(64\sqrt{3}-18\right)t, 64t-16t^2+192\right\rangle$.

11c We want to maximize $y(t)$, so we find $\frac{dy}{dt}=64-32t$ and set it equal to zero, finding $t=2$. Evaluating $y(2)$ gives $64(2)-32(2)^2+192=256$ ft.

11d To begin, find the time at which the particle lands by setting $y(t)=0$; that is, $64t-16t^2+192=0$. This gives $t\in\{-2,6\}$, from which we can immediately discard $t=-2$ because it is nonsensical to have negative time given the situation. Therefore the particle hits the ground at $t=6$ s; we find its horizontal displacement by finding $x(6)=\left(64\sqrt{3}-18\right)(6)=384\sqrt{3}-108$ ft.

12a The acceleration vector $\mathbf{a}(t)$ is given by $\mathbf{a}(t)=\left\langle\frac{d^2x}{dt^2},\frac{d^2y}{dt^2}\right\rangle=\left\langle 12-6t,\frac{4(t-4)^3}{1+(t-4)^4}\right\rangle$. Evaluating this at $t=2$ gives $\mathbf{a}(2)=\left\langle 0,-\frac{32}{17}\right\rangle$. The speed at $t=2$ is given by the magnitude of the velocity vector at $t=2$; since the velocity vector is $\mathbf{v}(t)=\left\langle\frac{dx}{dt},\frac{dy}{dt}\right\rangle=\left\langle 12t-3t^2,\ln\left(1+(t-4)^4\right)\right\rangle$, we have $\mathbf{v}(2)=\langle 12,\ln 17\rangle$, so $\|\mathbf{v}(2)\|=\sqrt{12^2+(\ln 17)^2}$ ≈ 12.33.

12b The change in the object's y-coordinate is given by integrating $\frac{dy}{dt}$ with respect to time over the time interval in question; this is $\int_0^2 \ln\left(1+(t-4)^4\right)dt$. The integrand cannot be antidifferentiated without the techniques of complex analysis, but the calculator will (after rather a while) give the approximation 8.671. However, note that this is the *change* in the object's y-coordinate; to get the actual y-coordinate, we must add its initial y-coordinate of 5, giving 13.671.

12c We find the slope at P by $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{\ln\left(1+(t-4)^4\right)}{12t-3t^2}$; at $t=2$, $\frac{dy}{dx}=\frac{\ln\left(1+(2-4)^4\right)}{12(2)-3(2)^2}=\frac{\ln 17}{12}$. Since we know the x-coordinate of P (it is given) and the y-coordinate (from part **b**), we can write the equation for the line in point-slope form as $y-13.671=\frac{\ln 17}{12}(x-3)$.

12d The object is stationary in the x-direction when $\frac{dx}{dt}=0$; that is, $12t-3t^2=0$, or $t\in\{0,4\}$. The object is stationary in the y-direction when $\frac{dy}{dt}=0$; that is, $\ln\left(1+(t-4)^4\right)=0$, or $t=4$. The object is wholly stationary at the intersection of the solution sets, which is $t=4$.