AP Calculus BC
Review — Chapter 12 (Sequences and Series), Part Two

Things to Know and Be Able to Do

- Understand the meaning of a power series centered at either 0 or an arbitrary *a*
- Understand radii and intervals of convergence, and know how to find each
- Understand that functions can be represented by power series, how to do this, and why it might be desirable
- Perform operations on functions represented by power series
- Understand Taylor series (and Maclaurin series), be able to find their expansions to an arbitrary number of terms and in the general case, and understand Taylor's Theorem for the remainder estimate

Practice Problems

These problems should be done without a calculator. The original test, of course, required that you show relevant work for free-response problems.

1 The Taylor series about $x=5$ for a certain function *f* converges to $f(x)$ for all *x* in the interval of convergence. The n^{th} derivative of *f* at $x=5$ is given by $\left.\dfrac{d^n f}{dx^n}\right|_{x=5}=\dfrac{(-1)^n n!}{2^n(n+2)}$, and $f(5)=\dfrac{1}{2}$.

a Write the third-degree Taylor polynomial for *f* about $x=5$.

b Show that the sixth-degree Taylor polynomial for *f* about $x=5$ approximates $f(6)$ with error less than $\dfrac{1}{1000}$.

2 Find the sum of the series $\dfrac{\pi^2}{4^2}-\dfrac{\pi^4}{4^4 3!}+\dfrac{\pi^6}{4^6 5!}-\cdots+\dfrac{(-1)^k\pi^{2k+2}}{4^{2k+2}(2k+1)!}+\cdots$. Justify your answer.

3 Let *f* be given by $f(x)=\sin\left(5x+\dfrac{\pi}{4}\right)$, and let $P(x)$ be the third-degree Taylor polynomial for *f* about $x=0$.

a Find $P(x)$.

b Find the coefficient of x^{22} in the Taylor series for *f* about $x=0$.

c Use the Lagrange error bound to show that $\left|f\left(\tfrac{1}{10}\right)-P\left(\tfrac{1}{10}\right)\right|<\dfrac{1}{100}$.

d Let G be given by $G(x)=\int_0^x f(t)\,dt$. Write the third-degree Taylor polynomial for G about $x=0$.

4 A function *f* is defined by $f(x)=\dfrac{1}{3}+\dfrac{2}{3^2}x+\dfrac{3}{3^3}x^2+\cdots+\dfrac{n+1}{3^{n+1}}x^n+\cdots$ for all *x* in the interval of convergence of the given power series.

a Find the interval of convergence for this power series. Show the work that leads to your answer.

b Find $\lim\limits_{x\to 0}\dfrac{f(x)-\frac{1}{3}}{x}$.

c Write the first three nonzero terms and the general term for an infinite series that represents $\int_0^1 f(x)\,dx$.

d Find the sum of the series determined in part **c**.

5 For what values of x does the series $\sum_{n=1}^{\infty}\frac{(x-2)^n}{3^n n}$ converge?

a $-3\le x\le 3$ **b** $-3<x<3$ **c** $-1<x\le 5$ **d** $-1\le x\le 5$ **e** $-1\le x<5$

6 The first four terms of the Maclaurin series for $f(x)=\sqrt{1+x}$ are

a $1+\frac{x}{2}-\frac{x^2}{4}+\frac{3x^3}{8}$ **b** $1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}$ **c** $1-\frac{x}{2}+\frac{x^2}{8}-\frac{x^3}{16}$

d $1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{8}$ **e** $1-\frac{x}{2}+\frac{x^2}{4}-\frac{3x^3}{8}$

7 The coefficient of $\left(x-\frac{\pi}{4}\right)^3$ in the Taylor series about $x=\frac{\pi}{4}$ for $f(x)=\cos x$ is

a $-\frac{\sqrt{2}}{12}$ **b** $-\frac{1}{12}$ **c** $\frac{1}{12}$ **d** $\frac{1}{6\sqrt{2}}$ **e** $-\frac{1}{3\sqrt{2}}$

8 The interval of convergence of $\sum_{n=0}^{\infty}\frac{(x-1)^n}{3^n}$ is

a $-3<x\le 3$ **b** $-3\le x\le 3$ **c** $-2<x<4$ **d** $-2\le x<4$ **e** $0\le x\le 2$

9 The coefficient of x^4 in the Maclaurin series for $f(x)=e^{-x/2}$ is

a $-\frac{1}{24}$ **b** $\frac{1}{24}$ **c** $\frac{1}{96}$ **d** $-\frac{1}{384}$ **e** $\frac{1}{384}$

10 For what integer k, $k>1$, will both $\sum_{n=1}^{\infty}\frac{(-1)^{kn}}{n}$ and $\sum_{n=1}^{\infty}\left(\frac{k}{4}\right)^n$ converge?

a 6 **b** 5 **c** 4 **d** 3 **e** 2

Answers

1a $\frac{1}{2}-\frac{1}{6}(x-5)+\frac{1}{16}(x-5)^2-\frac{1}{40}(x-5)^3$

2 $\dfrac{\pi}{4\sqrt{2}}$

3a $P(x)=\dfrac{1}{\sqrt{2}}+\dfrac{5}{\sqrt{2}}x-\dfrac{25}{2\sqrt{2}}x^2-\dfrac{125}{6\sqrt{2}}x^3$

3b $-\dfrac{5^{22}}{22!\sqrt{2}}$ **3d** $G(x)=\frac{1}{\sqrt{2}}x+\frac{5}{2\sqrt{2}}x^2-\frac{25}{6\sqrt{2}}x^3$

4a $(-3,3)$ **4b** $\frac{2}{9}$

4c $\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+L+\frac{1}{3^{n+1}}+L$

4d $\frac{1}{2}$

5 e, **6** b, **7** a, **8** c, **9** e, **10** d

Solutions

1a $P_3(x)=\dfrac{1}{2}+\left.\dfrac{\frac{(-1)^n n!}{2^n(n+2)}}{n!}\right|_{n=1}(x-5)+\left.\dfrac{\frac{(-1)^n n!}{2^n(n+2)}}{n!}\right|_{n=2}(x-5)^2+\left.\dfrac{\frac{(-1)^n n!}{2^n(n+2)}}{n!}\right|_{n=3}(x-5)^3=\dfrac{1}{2}-\dfrac{1}{6}(x-5)+\dfrac{1}{16}(x-5)^2-\dfrac{1}{40}(x-5)^3$

1b The coefficient of the next (seventh-degree) term of the Taylor series about $x=5$ is $\left.\dfrac{\frac{(-1)^n n!}{2^n(n+2)}}{n!}\right|_{n=7}=-\dfrac{1}{1152}<\dfrac{1}{1000}$.

2 Recall that the Maclaurin series for $\sin x$ is $\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!}$ and note that this series' kth term can be rewritten as $\left(\dfrac{\pi}{4}\right)\dfrac{(-1)^k\left(\frac{\pi}{4}\right)^{2k+1}}{(2k+1)!}$. Ignore for a moment that the summand is multiplied by $\pi/4$; we will deal with that later. Now the summand has a very similar form to the series for $\sin x$: an alternating factor $(-1)^k$, a constant raised to $2k+1$, and a denominator of $(2k+1)!$. Since the constant that is raised to the $(2k+1)$th power is $\pi/4$, we posit that $\pi/4$ is the argument for the sine function. Since $\sin(\pi/4)=\dfrac{1}{\sqrt{2}}$, that is the series' sum. Don't forget about the factor of $\pi/4$ from the beginning; the sum of the original series is $(\pi/4)\left(1/\sqrt{2}\right)=\dfrac{\pi}{4\sqrt{2}}$.

3a $f(0)=\dfrac{1}{\sqrt{2}}$, $f'(0)=\dfrac{5}{\sqrt{2}}$, $f''(0)=-\dfrac{25}{\sqrt{2}}$, and $f'''(0)=-\dfrac{125}{\sqrt{2}}$. Dividing each of these by the derivative's order's factorial gives, respectively, $\dfrac{1}{\sqrt{2}}$, $\dfrac{5}{\sqrt{2}}$, $-\dfrac{25\sqrt{2}}{4}$, and $-\dfrac{125\sqrt{2}}{12}$. These are the coefficients of $P(x)$, so $P(x)=\dfrac{1}{\sqrt{2}}+\dfrac{5}{\sqrt{2}}x-\dfrac{25\sqrt{2}}{4}x^2-\dfrac{125\sqrt{2}}{12}x^3$.

3b There will be a factor of $\dfrac{1}{\sqrt{2}}$ in the answer because evaluating either $\cos(5x+\pi/4)\big|_{x=0}$ or $\sin(5x+\pi/4)\big|_{x=0}$ will give $\cos(\pi/4)=\sin(\pi/4)=\dfrac{1}{\sqrt{2}}$. The Chain Rule will have to be applied 22 times, so we also have a factor of 5^{22}. Due to alternating signs in the power series for the sine function, the 22nd power's term will be negative, and we must also divide by $22!$. Therefore the coefficient is $-\dfrac{5^{22}}{22!\sqrt{2}}$.

3c Note that $\left|f(1/10)-P(1/10)\right|\le\frac{M}{4!}\left(\frac{1}{10}\right)^4$ where M is an upper bound on $f^4(x)$ on $[0,1/10]$. We have $M=625$, so the error is $\varepsilon\le\frac{625}{4!}\left(\frac{1}{10}\right)^4=\frac{1}{384}<\frac{1}{100}$. (The formula for the Lagrange error bound, found on page 799 of the textbook, is $\left|R_n(x)\right|\le\frac{M}{(n+1)!}|x-a|^{n+1}$ where M is as described, a is the center of the power series expansion, and n is the order of the power series.

3d Antidifferentiate each term of $P(x)$ to get $G(x)=\frac{x}{\sqrt{2}}+\frac{5\sqrt{2}}{4}x^2-\frac{25\sqrt{2}}{12}x^3-\frac{125\sqrt{2}}{48}x^4$. The last term may be ignored because the problem requires only the third-degree Taylor polynomial.

4a Letting $a_n=\frac{n+1}{3^{n+1}}x^n$, consider $\left|\frac{a_{n+1}}{a_n}\right|=\left|\frac{\frac{n+2}{3^{n+2}}x^{n+2}}{\frac{n+1}{3^{n+1}}x^{n+1}}\right|=\left|\frac{n+2}{3(n+1)}\cdot\frac{x\cdot x^{n+1}}{x^{n+1}}\right|=|x/3|\left(\frac{n+2}{n+1}\right)$. Thus we are interested in $|x/3|<1$ so $|x|<3$, meaning that the radius of convergence R is $R=3$. We must now test the endpoints, -3 and 3: if $x=-3$, then we have $\sum_{n=1}^{\infty}\frac{n+1}{3^{n+1}}(-3)^n=\frac{1}{3}\sum_{n=1}^{\infty}(-1)^n(n+1)$. This does not converge, and it can be similarly shown that when $x=3$ the series also does not converge. Therefore the interval of convergence I is $I=(-3,3)$.

4b $$\lim_{x\to 0}\frac{f(x)-\frac{1}{3}}{x}=\lim_{x\to 0}\frac{\left(\frac{1}{3}+\frac{2}{3^2}x+\frac{3}{3^3}x^2+\cdots+\frac{n+1}{3^{n+1}}x^n+\cdots\right)-\frac{1}{3}}{x}=\lim_{x\to 0}\frac{\frac{2}{3^2}x+\frac{3}{3^3}x^2+\cdots+\frac{n+1}{3^{n+1}}x^n+\cdots}{x}=\lim_{x\to 0}\left(\frac{2}{3^2}+\frac{3}{3^3}x+\cdots+\frac{n+1}{3^{n+1}}x^{n-1}+\cdots\right)=\frac{2}{3^2}=\frac{2}{9}.$$

4c Antidifferentiating each term of $f(x)$ gives $\int f(x)dx=\frac{1}{3}x+\frac{1}{9}x^2+\frac{1}{27}x^3+\cdots+\left(\frac{x}{3}\right)^{n+1}+\cdots$. Evaluating this at $x=1$ (since all the terms for $x=0$ are themselves zero) gives $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots+\frac{1}{3^{n+1}}+\cdots$.

4d This is a geometric series with first term $1/3$ and ratio $1/3$, so its sum is $\int_0^1 f(x)dx=\frac{1/3}{1-1/3}=\frac{1/3}{2/3}=\frac{1}{2}$.

5 We can immediately eliminate choices **a** and **b**; since the series is centered at $x=2$, so must the interval of convergence be. It will suffice to test the endpoints $x=-1$ and $x=5$. At $x=-1$, the series is $\sum_{n=1}^{\infty}\frac{(-3)^n}{3^n n}=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ which converges by the Alternating Series Test. At $x=5$, the series is $\sum_{n=1}^{\infty}\frac{3^n}{3^n n}=\sum_{n=1}^{\infty}\frac{1}{n}$ which is the harmonic series, so it diverges. Therefore the interval is $-1\le x<5$, choice **e**.

6 $f(0)=1$, $f'(0)=1/2$, $f''(0)=-1/4$, and $f'''(0)=3/8$. Dividing by the derivatives' orders' factorials gives, respectively, 1, $1/2$, $-1/8$, and $1/16$, so the first four terms are $1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}$, which is **b**.

7 Several derivatives of $f(x)=\cos x$ at $x=\pi/4$ are $f'(\pi/4)=-1/\sqrt{2}$, $f''(\pi/4)=-1/\sqrt{2}$, and $f'''(\pi/4)=1/\sqrt{2}$. The last one just given is the relevant derivative, but we must first divide by 3! to get the correct coefficient. Thus we are interested in $\frac{1/\sqrt{2}}{3!}=\frac{1/\sqrt{2}}{6}=\frac{1}{6\sqrt{2}}$, option **d**.

8 As in problem **5**, we can eliminate **a** and **b** because the interval of convergence must be symmetric around the center point, $x = 1$. We find the radius of convergence by considering $\left| \frac{(x-1)^{n+1}/3^{n+1}}{(x-1)^n/3^n} \right| = \left| \frac{x-1}{3} \right|$, so we are interested in $\left| \frac{x-1}{3} \right| < 1$ or $|x-1| < 3$, meaning the radius of convergence is 3. Therefore **e** is wrong, because it involves a radius of 1. Given the two remaining options, we need only test the endpoint $x = -2$ because both indicate that $x = 4$ is divergent. So we look at $\sum_{n=0}^{\infty} \frac{(-3)^n}{3^n} = \sum_{n=0}^{\infty} (-1)^n$ which diverges. The interval is therefore $-2 < x < 4$, choice **c**.

9 Several derivatives of $f(x)$ at $x = 0$ are $f'(0) = -1/2$, $f''(0) = 1/4$, $f'''(0) = -1/8$, and $f^4(0) = 1/16$. The last one just given is relevant, but we must divide by 4! To get $f^4(0) = \frac{1/16}{4!} = \frac{1/16}{24} = \frac{1}{384}$, choice **e**.

10 We can immediately eliminate **a**, **b**, and **c** because these would produce in the second series a geometric series with a ratio too large to allow for convergence. **e**, too, is impossible because the first series would then have a factor of $(-1)^{2n}$ which for integer *n* is always even and thus always 1. That, then, is the harmonic series; the harmonic series is known to diverge. Therefore **d** is the correct answer.