Multivariable Calculus
Review Problems — Chapter 13

Things to Know and Be Able to Do

- Understand, and analyze equations of surfaces and curves in two and three dimensions given parametrically
- Work with scalar and vector projections, understanding what they mean and how they can be used
- Use Cartesian, cylindrical, and spherical coordinates and convert both points and surfaces between them
- Analyze the relationships between figures in two- and three-dimensional space

Practice Problems

These problems should be done without a calculator. The original test, of course, required that you show relevant work.

1 Write equations or inequalities to describe the following loci:

1a the *interior* of the sphere with radius 4 centered at $(2,4,-1)$

1b the plane through $(3,5,2)$ that is perpendicular to the x-axis

1c the circle with radius 2 centered at $(1,3,2)$ lying in the plane $y=3$

1d the set of points in space equidistant from the origin and $(0,2,4)$

1e the set of points in space equidistant from $(0,0,4)$ and the plane $z=-2$

2 Find the equation of the line of intersection of the planes $4x-3y+z=2$ and $2x+5y-3z=-4$.

3 Given the three points $A=(0,-1,2)$, $B=(2,-3,1)$, and $C=(-2,4,-1)$, do the following, giving exact answers:

3a Find parametric equations for the line that passes through A and C

3b Find the scalar projection of $\overrightarrow{AC}$ onto $\overrightarrow{AB}$.

3c Write $\overrightarrow{AC}$ as the sum of two vectors, one parallel to $\overrightarrow{AB}$ and one perpendicular to $\overrightarrow{AB}$.

3d Find the equation of the plane containing A, B, and C.

4 Consider the point in spherical coordinates given by $(\rho,\theta,\varphi)=(12,5\pi/4,\pi/3)$. Find the exact coordinates of this point in both rectangular and cylindrical coordinates.

5 Change the equations for these surfaces into Cartesian coordinates, and give a brief description of the surface.

5a $\theta=\pi/2$ **5b** $\rho=4\csc\varphi$ **5c** $r=4\cos\theta$

6 The equation of a surface in $\mathbb{R}^3$ is given in spherical coordinates by $\rho=8\cos\varphi$ for $\varphi\in[0,\pi/6]$.

6a Give the equation in rectangular coordinates. Include restrictions on coordinates if necessary.

6b Give the equation in cylindrical coordinates. Include restrictions on coordinates if necessary.

6c Give a name or description of this surface. Be as specific as possible.

7 Consider a point P on a line $\mathbf{r}$ determined by point S and direction vector $\mathbf{v}$. Then $(PS\times\mathbf{v})\times\mathbf{v}$ gives a vector along which the distance from P to $\mathbf{r}$ is measured.

7a Explain geometrically or algebraically why the vector $(PS\times\mathbf{v})\times\mathbf{v}$ is in the correct direction to measure the distance from the point to the line.

7b Write a vector expression that represents the directed distance from P to $\mathbf{r}$ as a projection.

Bonus Problem Show that the lines $\frac{x-1}{1}=\frac{y-2}{1}=\frac{z-3}{2}$ and $\frac{x-1}{-1}=\frac{y-2}{-1}=\frac{z-3}{-2}$ lie entirely on the surface of the hyperbolic paraboloid given by $z=y^2-x^2$.

Answers

1a $(x-2)^2+(y-4)^2+(z+1)^2<16$

1b $x=3$

1c $\begin{cases}(x-1)^2+(z-2)^2=4\\ y=3\end{cases}$

1d $y+2z=5$

1e $x^2+y^2-12z+12=0$

2 answers may vary; the line's direction vector is parallel to $\langle 2,7,13\rangle$

3a answers may vary; the line's direction vector is parallel to $\langle -2,5,-3\rangle$

3b $-11/3$

3c $\langle -22/9,22/9,11/9\rangle+\langle 4/9,23/9,-38/9\rangle$

3d $11x+8y+6z=4$

4 $\left(6\sqrt{3},5\pi/4,6\right)$ and $\left(-3\sqrt{6},-3\sqrt{6},6\right)$

5a $x=0,$ which is the *yz*-plane for positive *y*

5b $x^2+y^2=16,$ a right circular cylinder centered about the *z*-axis with radius 4

5c $(x-2)^2+y^2=4,$ a right circular cylinder tangent to the *z*-axis with radius 2

6a $x^2+y^2+z^2=8z$ for $z\ge 6$

6b $r^2+z^2=8z$ for $z\ge 6$

6c part of a sphere centered at $(0,0,4)$ with radius 4

7b $\dfrac{((PS\times\mathbf{v})\times\mathbf{v})\cdot PS}{\|(PS\times\mathbf{v})\times\mathbf{v}\|^2}((PS\times\mathbf{v})\times\mathbf{v})$

Solutions

1a–d These are essentially questions of rote. If you have trouble with them, please see your teacher.

1e Consider an arbitrary point (x,y,z). That point's distance from $(0,0,4)$ is given by $\sqrt{x^2+y^2+(z-4)^2}$. Since the formula for the distance D between a point (x_0,y_0,z_0) and a plane given by $ax+by+cz+d=0$ is $D=\dfrac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$, the distance from our arbitrary (x,y,z) to the given plane is $\dfrac{|0x+0y+z+2|}{\sqrt{0^2+0^2+1^2}}=|z+2|$. This means that we are interested in $\sqrt{x^2+y^2+(z-4)^2}=|z+2|$. Squaring both sides yields $x^2+y^2+(z-4)^2=z^2+4z+4,$ which can be rewritten as $x^2+y^2-12z+12=0.$

2 The planes' normal vectors are, respectively, $\mathbf{n}_1=\langle 4,-3,1\rangle$ and $\mathbf{n}_2=\langle 2,5,-3\rangle$. The line of intersection has a direction vector perpendicular to both of these, which we can find by taking $\mathbf{n}_1\times\mathbf{n}_2=\langle 4,14,26\rangle$. It is equivalent to state that as $\langle 2,7,13\rangle$ (dividing by two). Now it only remains to find a point shared by both planes. We can choose $z=0$ to get the system $\begin{cases}4x-3y+0=2\\ 2x+4y-3(0)=-4\end{cases}$ which solves to $(x,y,z)=(-2/11,-10/11,0)$, which is a point on the line of intersection. Combining that with the line's known direction vector means that the line can be given as $\langle x,y,z\rangle=\langle -2/11,-10/11,0\rangle+\langle 2,7,13\rangle t$.

3a $\overrightarrow{AC}$ is given by $C-A=(-2,4,-1)-(0,-1,2)=\langle -2,5,-3\rangle$. We must now find a point on that line; it is most convenient to use either A or C. Using A yields $\langle x,y,z\rangle=(0,-1,2)+\langle -2,5,-3\rangle t$.

3b Recall that the scalar projection of some **b** onto some **a** is $\text{comp}_{\mathbf{a}}\,\mathbf{b}=\dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}$. $\overrightarrow{AC}$ is given by $\langle -2,5,-3\rangle$, as we saw in **3a**, and $\overrightarrow{AB}$ is similarly given by $\langle 2,-2,-1\rangle$. Its magnitude is $\|\overrightarrow{AB}\|=\sqrt{2^2+(-2)^2+1^2}=\sqrt{9}=3.$ $\overrightarrow{AB}\cdot\overrightarrow{AC}=2(-2)+(-2)(5)+(-1)(-3)=-11,$ so $\text{comp}_{\overrightarrow{AB}}\,\overrightarrow{AC}=\dfrac{-11}{3}$.

3c The part parallel to $\overrightarrow{AB}$ is the projection of $\overrightarrow{AC}$ onto $\overrightarrow{AB}$, which is given by $\operatorname{proj}_{\overrightarrow{AB}} \overrightarrow{AC} = \left(\operatorname{comp}_{\overrightarrow{AB}} \overrightarrow{AC}\right) \dfrac{\overrightarrow{AB}}{\left\|\overrightarrow{AB}\right\|}$.

This is $-\dfrac{11}{3}\left(\dfrac{\langle 2,-2,-1\rangle}{3}\right) = -\dfrac{11}{9}\langle 2,-2,-1\rangle = \langle -22/9, 22/9, 11/9\rangle$. The part perpendicular to $\overrightarrow{AB}$ can be found simply by subtracting the already-found part from $\overrightarrow{AC}$ since the two add to $\overrightarrow{AC}$: this is $\langle -2,5,-3\rangle - \langle -22/9, 22/9, 11/9\rangle = \langle 4/9, 23/9, -38/9\rangle$. So $\overrightarrow{AC}$ can be written as $\langle -22/9, 22/9, 11/9\rangle + \langle 4/9, 23/9, -38/9\rangle$.

3d Consider $\overrightarrow{AB}$ and $\overrightarrow{AC}$. The normal vector to the plane is normal to both of these vectors, so it can be found by taking their cross product: $\overrightarrow{AB} \times \overrightarrow{AC} = \langle 11,8,6\rangle$. Therefore the plane is given by $11x + 8y + 6z = d$ for some d; to find it, plug in the point A which must be contained within the plane: $11(0) + 8(-1) + 6(2) = d \Rightarrow d = 4$. So the plane we are looking for is $11x + 8y + 6z = 4$.

4 A diagram is vital for this problem! Draw a diagram and you will find that $x = \rho \sin\varphi \cos\theta$, $y = \rho \sin\varphi \sin\theta$, and $z = \rho\cos\varphi$. The Cartesian coordinates are therefore given by $(x,y,z) = \left(12\sin\frac{\pi}{3}\cos\frac{5\pi}{4}, 12\sin\frac{\pi}{3}\sin\frac{5\pi}{4}, 12\cos\frac{\pi}{3}\right)$ $= \left(-3\sqrt{6}, -3\sqrt{6}, 6\right)$. Now it is probably easiest to convert these Cartesian coordinates into cylindrical; remember that the z-coordinates are the same, and the polar θ-coordinate is the same as the spherical θ-coordinate. Therefore we already have $(r, 5\pi/4, 6)$ for the cylindrical coordinates, and $r = \sqrt{x^2 + y^2} = \sqrt{\left(-3\sqrt{6}\right)^2 + \left(-3\sqrt{6}\right)^2}$ $= \sqrt{54 + 54} = \sqrt{108} = 6\sqrt{3}$. So the cylindrical coordinates are $(r,\theta,z) = \left(6\sqrt{3}, 5\pi/4, 6\right)$.

5a This too is essentially a question of rote. If you have trouble with it, please see your teacher.

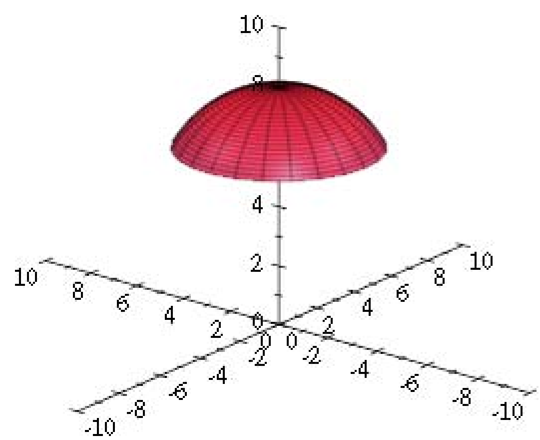
5b Write this as $\rho = \dfrac{4}{\sin\varphi}$, so $\rho\sin\varphi = 4$. Since $\rho\sin\varphi = r$ (you need a diagram!), this means that $r = 4$. This should be recognizable as a right circular cylinder centered about the z-axis with radius 4. To put it into Cartesian coordinates, $\sqrt{x^2 + y^2} = 4$ or $x^2 + y^2 = 16$.

5c Multiply both sides by r to get $r^2 = 4r\cos\theta$. This means $x^2 + y^2 = 4x$, on which we can complete the square: $x^2 - 4x + 4 + y^2 = 4$ so $(x-2)^2 + y^2 = 4$. Since z is not involved in the equation, it can be anything; treat this if you like as an equation in $\mathbb{R}^2$ and then "slide it up and down"; it's a circle that gets slid to become a right circular cylinder. Its radius is $\sqrt{4} = 2$, and since it is centered at $(2,0,z)\,\forall z$, it is tangent to the z-axis.

6a Multiply both sides by ρ: $\rho^2 = 8\rho\cos\varphi$, and as we have previously noted, $\rho\cos\varphi = z$. Also, $\rho^2 = x^2 + y^2 + z^2$. Thus $x^2 + y^2 + z^2 = 8z$. A diagram showing the restrictions is included at right. From the φ restriction, we get that $z \ge 6$.

6b Since $r^2 = x^2 + y^2$, this is $r^2 + z^2 = 8z$, and the restriction $z \ge 6$ still holds.

6c The diagram is shown at right; this is a portion of a sphere. Completing the square from the Cartesian equation gives $x^2 + y^2 + (z-4)^2 = 4^2$, so the sphere's center is $(0,0,4)$ and its radius is 4.



7a Note that $\overrightarrow{PS}\times\mathbf{v}$ is perpendicular to both $\overrightarrow{PS}$ and $\mathbf{v}$, and $\left(\overrightarrow{PS}\times\mathbf{v}\right)\times\mathbf{v}$ is perpendicular to both $\left(\overrightarrow{PS}\times\mathbf{v}\right)$ and $\mathbf{v}$. Thus $\overrightarrow{PS}$ and $\left(\overrightarrow{PS}\times\mathbf{v}\right)\times\mathbf{v}$ determine a plane perpendicular to **r** that contains *P*, and thus the perpendicular line from *P* to **r** is in that plane. Draw a diagram of the situation to convince yourself of this.

7b The directed distance in question is given by $\mathbf{d}=\left\|\operatorname{proj}_{\left(\overrightarrow{PS}\times\mathbf{v}\right)\times\mathbf{v}}\overrightarrow{PS}\right\|$, which is $\dfrac{\left(\left(\overrightarrow{PS}\times\mathbf{v}\right)\times\mathbf{v}\right)\cdot\overrightarrow{PS}}{\left\|\left(\overrightarrow{PS}\times\mathbf{v}\right)\times\mathbf{v}\right\|^2}\left(\left(\overrightarrow{PS}\times\mathbf{v}\right)\times\mathbf{v}\right)$.

Bonus Problem These lines can be written parametrically as $\ell_1:\langle x,y,z\rangle=(1,2,3)+\langle 1,1,2\rangle t$ and $\ell_2:\langle x,y,z\rangle=(1,2,3)+(-1,-1,-2)t$, respectively. Now we plug each of these in turn into the hyperbolic paraboloid's equation: for ℓ_1, $3+2t=(2+t)^2-(1+t)^2$, and for ℓ_2, $3-2t=(2-t)^2-(1-t)^2$. Expanding the first one, $3+2t=\left(4+4t+t^2\right)-\left(1+2t+t^2\right)$, or $3+2t=3+2t$, which is a true statement. Therefore this line lies entirely on the hyperbolic paraboloid. Similarly, expansion of the second gives $3-2t=\left(4-4t+t^2\right)-\left(1-2t+t^2\right)$, or $3-2t=3-2t$. This, too, is true, so the second line also lies entirely on the surface of the paraboloid.