Multivariable Calculus
Review Problems — Chapter 14

Things to Know and Be Able to Do

- Analyze curves in $\mathbb{R}^3$ given by vector-valued functions, finding their shape, value, tangent vector function, normal vector function, binormal vector function, arc length, curvature, torsion, osculating circle, derivatives, integral
- Parameterize space curves in terms of arc length (and otherwise) and know why this is useful
- Understand the meaning of velocity, acceleration, and their components with respect to space curves
- Determine the intersections of surfaces in terms of space curves

Practice Problems

You may use a calculator to evaluate derivatives and integrals and simplify expressions. The original test, of course, required that you show relevant work.

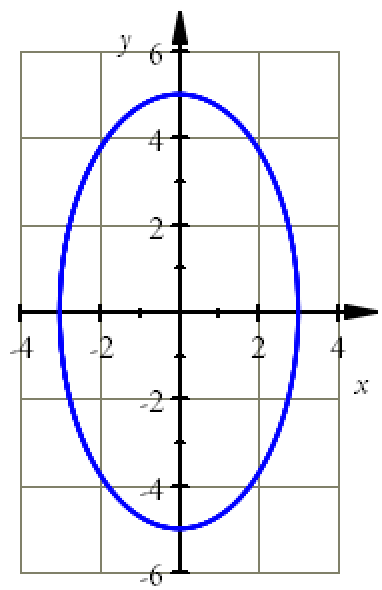
1 Find parametric equations for the *line* tangent to the curve $\mathbf{r}(t)=e^t\hat{\mathbf{i}}+(\sin t)\hat{\mathbf{j}}+\ln(1-t)\hat{\mathbf{k}}$ at $t=0$.

2 Find the *point* on the curve $\mathbf{r}(t)=(5\sin t)\hat{\mathbf{i}}+(5\cos t)\hat{\mathbf{j}}+12t\hat{\mathbf{k}}$ at a distance of $13\pi/4$ along the curve from the origin in the direction of increasing arc length.

3 Consider the curve $\mathbf{r}(t)=t\hat{\mathbf{i}}+\left(t^3+3t^2-24t\right)\hat{\mathbf{j}}$.

3a Find the velocity and acceleration functions $\mathbf{v}(t)$ and $\mathbf{a}(t)$ for this vector-valued function.
3b Find the speed of the particle at $t=2$.
3c Give the unit tangent vector and unit normal vector at $t=2$.
3d Find the curvature κ at $t=2$.
3e Find a_T, the scalar component of acceleration in the direction of the tangent, at $t=2$.

4 Find the equation for the osculating circle of the ellipse $x^2/9+y^2/25=1$ at the point $(0,-5)$ and draw the osculating circle on the graph given. (Hint: write the ellipse in parametric form first. The osculating circle has the same curvature, tangent vector, and normal vector at the point.)



5 Consider the curve $\mathbf{r}(t)=\langle 3\cosh(2t), 3\sinh(2t), 6t\rangle$. Find each of the following at the point where $t=\ln 2$:

5a the unit tangent vector $\hat{\mathbf{T}}$
5b the unit normal vector $\hat{\mathbf{N}}$
5c the unit binormal vector $\hat{\mathbf{B}}$
5d the curvature κ
5e the torsion τ, using the formula $\tau=\dfrac{\begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix}}{\|\mathbf{v}\times\mathbf{a}\|^2}$

6 Although we use a different formula most of the time, torsion τ is defined by $\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}$, where s is the parameter for arc length.

6a Show that $\frac{d\mathbf{B}}{ds} = \mathbf{T} \times \frac{d\mathbf{N}}{ds}$. (Hint: start with a definition for **B** and differentiate.)

6b Explain why $\frac{d\mathbf{B}}{ds} \parallel \mathbf{N}$.

6c Explain why the torsion of a plane curve must be zero.

7 The curve with parametric equations $\langle x, y, z \rangle = \langle \sin t, \cos t, \sin^2 t \rangle$ is the curve of intersection of a circular cylinder and a parabolic cylinder. Find Cartesian equations for the cylinders and use these equations to draw the curve.

Answers

1 $\langle x,y,z\rangle=(1,0,0)+\langle 1,1,-1\rangle t$

2 $\left(5/\sqrt{2},5/\sqrt{2},3\pi\right)$

3a $\mathbf{v}(t)=\left\langle 1,3t^2+6t-24\right\rangle$; $\mathbf{a}(t)=\langle 0,6t+6\rangle$

3b $\|\mathbf{v}(2)\|=1$

3c $\hat{\mathbf{T}}=\langle 1,0\rangle$; $\hat{\mathbf{N}}=\langle 0,1\rangle$

3d $\kappa(2)=18$

3e $a_T=0$

4 $x^2+\left(y+16/5\right)^2=\left(9/5\right)^2$

5a $\hat{\mathbf{T}}=\left\langle 15\sqrt{2}/34,1/\sqrt{2},4\sqrt{2}/17\right\rangle$

5b $\hat{\mathbf{N}}=\left\langle 8/17,0,-15/17\right\rangle$

5c $\hat{\mathbf{B}}=\left\langle -15\sqrt{2}/34,1/\sqrt{2},-4\sqrt{2}/17\right\rangle$

5d $\kappa=32/867$

5e $\tau=32/867$

7 $x^2+y^2=1$ and $z=x^2$

Solutions

1 We evaluate $\mathbf{r}(0)=\left\langle e^0,\sin 0,\ln 1\right\rangle=\langle 1,0,0\rangle$, which we will use as a point on our line. Finding the tangent line requires differentiating $\mathbf{r}(t)$ to get $\mathbf{r}'(t)=\left\langle e^t,\cos t,-\dfrac{1}{1-t}\right\rangle$, which at $t=0$ is $\mathbf{r}'(0)=\left\langle e^0,\cos 0,-\dfrac{1}{1-0}\right\rangle=\langle 1,1,-1\rangle$, which is a direction vector for the tangent line. Thus the line is $\langle x,y,z\rangle=(1,0,0)+\langle 1,1,-1\rangle t$.

2 Recall that arc length s is given by $s(t)=\int_a^t\|\mathbf{r}'(u)\|du$. Therefore we first evaluate $\|\mathbf{r}'(u)\|$ (we are using u because t is needed as the upper limit of the integral. This requires finding $\mathbf{r}'(u)=\langle 5\cos t,-5\sin t,12\rangle$, of which the magnitude is $\sqrt{(5\cos t)^2+(-5\sin t)^2+12^2}=\sqrt{25\cos^2 t+25\sin^2 t+144}=\sqrt{25(1)+144}=\sqrt{169}=13$. Since we want to solve for the point at which the arc length is a known value, we set $\int_0^t 13\,du=13\pi/4$, or $13u]_0^t=13\pi/4$, so $13t=13\pi/4$ and $t=\pi/4$. We still have to find the point, so we evaluate $\mathbf{r}(\pi/4)=\left\langle 5/\sqrt{2},5/\sqrt{2},3\pi\right\rangle$.

3a Since $\mathbf{v}(t)=\mathbf{r}'(t)$, we differentiate each component: $\mathbf{v}(t)=\left\langle 1,3t^2+6t-24\right\rangle$. Similarly, $\mathbf{a}(t)=\mathbf{v}(t)=\mathbf{r}''(t)$, so we differentiate each component again to get $\mathbf{a}(t)=\langle 0,6t+6\rangle$.

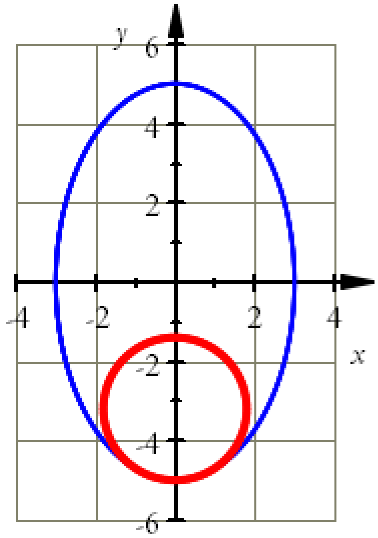
3b Speed is given by $\|\mathbf{v}(t)\|$, so we evaluate $\mathbf{v}(2)=\left\langle 1,3(2)^2+6(2)-24\right\rangle=\langle 1,0\rangle$; the magnitude of that is 1.

3c The unit tangent vector $\hat{\mathbf{T}}$ is given by $\hat{\mathbf{T}}(t)=\dfrac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|}$, so we must find $\|\mathbf{v}(t)\|$. This is $\sqrt{1^2+\left(3t^2+6t-24\right)^2}=\sqrt{9t^4+36t^3-108t^2-288t+577}$ which at $t=2$ is 1. So $\hat{\mathbf{T}}(2)=\dfrac{\langle 1,0\rangle}{1}=\langle 1,0\rangle$.

3d The curvature κ is given by $\kappa(t)=\dfrac{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}=\dfrac{\|\langle 0,0,6t+6\rangle\|}{\left(9t^4+36t^3-108t^2-288t+577\right)^{3/2}}$ which at $t=2$ is 18.

3e By definition, $a_T=\dfrac{d}{dt}\|\mathbf{v}\|$, which is $\dfrac{d}{dt}\left(\sqrt{9t^4+36t^3-108t^2-288t+577}\right)$. This is easier than it looks to do by hand, but can be evaluated by your calculator also as $\dfrac{18t^3+54t^2-108t-144}{\sqrt{9t^4+36t^3-108t^2-288t+577}}$, which at $t=2$ is 0.

4 The ellipse can be parameterized as $\mathbf{r}(t)=\langle 3\cos t,5\sin t\rangle$, in which case the point $(0,-5)$ is at $t=3\pi/2$. $\kappa(t)=\dfrac{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}=\dfrac{\|\langle -3\sin t,5\cos t\rangle\times\langle 3\cos t,-5\sin t\rangle\|}{\|\langle -3\sin t,5\cos t\rangle\|^3}$. It is simplest to evaluate this by calculator; at $t=3\pi/2$, $\kappa=5/9$. Since the radius of the osculating circle is the reciprocal of the curvature, $r=9/5$. The circle is known to touch $(0,-5)$ at its bottom, so knowing its radius also allows us to determine that the y-coordinate of the osculating circle's center is $-16/5$. Since the curve is symmetric about the y-axis, so must be the osculating circle, and thus its center has x-coordinate 0. To recapitulate, the circle has radius $9/5$ and is centered at $(0,-16/5)$, meaning that its equation is $x^2+(y+16/5)^2=(9/5)^2$. The picture is shown at right.



5a Again, $\hat{\mathbf{T}}(t)=\dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$, which means that $\hat{\mathbf{T}}(t)=\dfrac{\langle 6\sinh(2t),6\cosh(2t),6\rangle}{6\sqrt{2}\cosh(2t)}$ $=\left\langle \dfrac{\tanh(2t)}{\sqrt{2}},\dfrac{1}{\sqrt{2}},\dfrac{\operatorname{sech}(2t)}{\sqrt{2}}\right\rangle$. At $t=\ln 2$ this is $\langle 15\sqrt{2}/34,1/\sqrt{2},4\sqrt{2}/17\rangle$.

5b Since $\hat{\mathbf{N}}(t)=\dfrac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$, we have $\hat{\mathbf{N}}(t)=\dfrac{\langle\sqrt{2}\operatorname{sech}^2(2t),0,-\sqrt{2}\operatorname{sech}(2t)\tanh(2t)\rangle}{\sqrt{2}\operatorname{sech}(2t)}=\langle\operatorname{sech}(2t),0,-\tanh(2t)\rangle$ which at $t=\ln 2$ is $\langle 8/17,0,-15/17\rangle$.

5c Since $\hat{\mathbf{B}}(t)=\hat{\mathbf{T}}(t)\times\hat{\mathbf{N}}(t)$, we can easily determine that $\hat{\mathbf{B}}(\ln 2)=\langle 15\sqrt{2}/34,1/\sqrt{2},4\sqrt{2}/17\rangle\times\langle 8/17,0,-15/17\rangle$ $=\langle -15\sqrt{2}/34,1/\sqrt{2},-4\sqrt{2}/17\rangle$.

5d Another curvature formula is $\kappa(t)=\dfrac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$. We have already found each of these, so this is the most convenient formula to use; it gives $\kappa(t)=\dfrac{\sqrt{2}\operatorname{sech}(2t)}{6\sqrt{2}\cosh(2t)}=\dfrac{1}{6\cosh^2(2t)}$ which at $t=\ln 2$ is $32/867$.

5e Plugging in everything to the given formula gives $\tau=\dfrac{\begin{vmatrix}45/4 & 51/4 & 6\\ 51/2 & 45/2 & 0\\ 45 & 51 & 0\end{vmatrix}}{\left(153\sqrt{2}\right)^2}=\dfrac{1728}{46818}=\dfrac{32}{867}$.

6a Begin with $\mathbf{B}=\mathbf{T}\times\mathbf{N}$. Then $\dfrac{d\mathbf{B}}{ds}=\dfrac{d}{ds}(\mathbf{T}\times\mathbf{N})=\mathbf{T}\times\dfrac{d\mathbf{N}}{ds}+\dfrac{d\mathbf{T}}{ds}\times\mathbf{N}$ by the product rule. According to the Frenet-Serret formulas (page 906 of Stewart's textbook), $\dfrac{d\mathbf{T}}{ds}=\kappa\mathbf{N}$, so we can substitute in the second term: $\dfrac{d\mathbf{B}}{ds}=\mathbf{T}\times\dfrac{d\mathbf{N}}{ds}+\kappa\mathbf{N}\times\mathbf{N}$. Since the cross product of any vector with itself is always the zero vector, the second term is the zero vector and $\dfrac{d\mathbf{B}}{ds}=\mathbf{T}\times\dfrac{d\mathbf{N}}{ds}$, which is what we wanted.

6b It can be shown (exercise 49, section 14.3) that $\mathbf{T}\perp\dfrac{d\mathbf{B}}{ds}\perp\mathbf{B}$. Any vector perpendicular to both must be parallel to $\mathbf{N}$ because $\mathbf{B}=\mathbf{T}\times\mathbf{N}$.

6c Since $\hat{\mathbf{B}}$ is a constant unit vector perpendicular to the plane containing the curve, $\dfrac{d\mathbf{B}}{ds}=\mathbf{0}$, but since $\dfrac{d\mathbf{B}}{ds}=-\tau\mathbf{N}$ by the Frenet-Serret formulas, we know $-\tau\mathbf{N}=\mathbf{0}$. This means that either $\tau=0$ or $\mathbf{N}=\mathbf{0}$, the latter of which we know not to be the case. Therefore $\tau=0$.

7 From the *x*- and *y*-components of the curve, the projection of the curve onto the *yz*-plane is an origin-centered circle with radius 1. This suggests that the right circular cylinder is given by $x^2 + y^2 = 1$. Note also that $z = x^2$. This is the other cylinder. The cylinders and the curve are shown at right.