Multivariable Calculus
Review Problems — Chapter 15, part 1

Things to Know and Be Able to Do

- Understand functions of several variables given by their descriptions, tables, explicit formulas, and diagrams or graphs
- Understand and apply the concept of level curves
- Evaluate limits of functions of several variables and use the delta-epsilon technique to prove them, or use appropriate techniques to prove their nonexistence
- Understand and evaluate of partial derivatives of any order
- Find tangent planes to surfaces in $\mathbb{R}^3$ and use them to develop linear approximations to the surface
- Use the chain rule for functions of several variables, including those defined in terms of other functions of one or several variables

Practice Problems

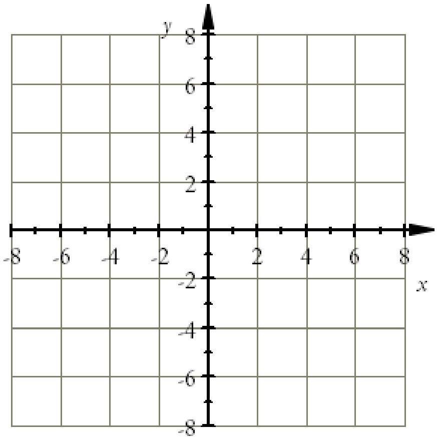
You may use a calculator. The original test, of course, required that you show relevant work.

1 Consider the real-valued function $z = f(x,y) = \sqrt{9x^2 + 4y^2 - 144}$.

1a On the given axes, sketch the domain of $f(x,y)$ as a region in $\mathbb{R}^2$.

1b Is the domain of the function open or closed? Explain.

1c Is the domain of the function bounded or unbounded? Explain.



2 Find $\left(\dfrac{\partial w}{\partial z}\right)_y$ if $w = e^{xy} + \dfrac{y}{z} - \sinh x$ subject to the constraint that $x^2 y - z\cos y + \ln(xz) = \tan y$.

3 For each of the following, determine if the limit exists. If it does, show algebraic work to evaluate the limit; if it does not, justify algebraically why not.

3a $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x+y}{x-y}$

3b $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$

4 Prove each of the following limits using the delta-epsilon definition.

4a $\displaystyle\lim_{(x,y)\to(-1,3)} (3x-2y) = -9$

4b $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^3+3x^2y}{x^2+y^2} = 0$

5a Find parametric equations for the line tangent to the curve of intersection of the surface $z = 2x^2 + 3y^2$ with the plane $y = 1$ at the point $(-2,1,11)$.

5b Find an equation for the plane tangent to the surface $z = 2x^2 + 3y^2$ at the point $(-2,1,11)$.

6 Consider $z = f(x,y)$, $x = g(t)$, and $y = h(r,t)$. Use the chain rule to write an expression that represents each of the following.

6a $\dfrac{\partial z}{\partial t}$

6b $\dfrac{\partial}{\partial t}\left(\dfrac{\partial z}{\partial y}\cdot\dfrac{\partial y}{\partial t}\right)$

7 The heat index I is a function of air temperature T and relative humidity H; that is, $I = f(T,H)$. Use the following table of values for each of the following.

Heat Index I [°F]

Relative Humidity H [%]

Air Temperature T [°F]

	50	55	60	65	70	75	80	85	90
90	96	98	100	103	106	109	112	115	119
92	100	103	105	108	112	115	119	123	128
94	104	107	111	114	118	122	127	132	137
96	109	113	116	121	125	130	135	141	146
98	114	118	123	127	133	138	144	150	157
100	119	124	129	135	141	147	154	161	168

7a Estimate the value of $\dfrac{\partial I}{\partial T}$ when $T = 94$°F and $H = 65\%$. Show your calculation.

7b Find a linear approximation for the heat index I when T is near 94°F and H is near 65%.

7c Use the linear approximation found in part **b** to estimate the heat index I when the temperature is 95°F and the relative humidity is 63%.

8 Use differentials to estimate the volume of wood required to make a hollow rectangular box whose inside measurements are $5\text{ ft} \times 3\text{ ft} \times 2\text{ ft}$ (length by width by depth) if the box is made from half-inch-thick lumber and the box has no top.

Bonus Problem Use differentials to give a rational number approximation for $\sqrt{26}\sqrt[3]{28}\sqrt[4]{17}$.

Answers

1b closed

1c unbounded

2 $\left(\frac{x(z\cos y-1)}{z(2x^2y+1)}\right)(ye^{xy}-\cosh x)-\frac{y}{z^2}$

3a the limit does not exist

3b 2

5a $\langle x,y,z\rangle=\langle -2+t,1,11-8t\rangle$

5b $-8x+6y-z-33=0$

6a $\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$

6b $\frac{\partial^2 z}{\partial y\partial x}\cdot\frac{dx}{dt}+\frac{\partial^2 x}{\partial y\partial t}\cdot\frac{\partial z}{\partial x}+\frac{\partial^2 z}{\partial y^2}\cdot\frac{\partial y}{\partial t}$

7a 13/4

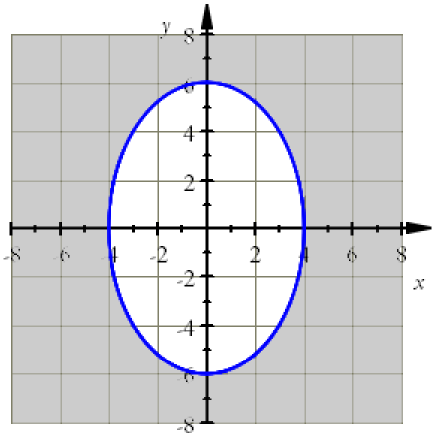
7b $I\approx 114+\tfrac{13}{4}(T-94°\mathrm{F})+\tfrac{7}{10}(H-65\%)$

7c 115.85°F

8 47/24 ft^3

Bonus Problem 6217/4320

Solutions



1a For the function to be real-valued, the radicand must be nonnegative; that is, $9x^2+4y^2-144\le 0$. Consider the case where $9x^2+4y^2-144=0$; this gives the ellipse shown on the diagram at right. Then either by inspection or by testing a point, we can show that the area including the ellipse and *outside* of it comprises the domain of the function. This is shaded in the diagram.

1b Recall that a set is defined as closed iff it contains all of its boundary points (in this case, the ellipse itself), and open otherwise. The inequality allows for equality, so the domain contains its entire boundary and is therefore closed.

1c Recall that a set is defined as bounded iff it can be contained within a disc of finite radius and unbounded if it cannot. Since the domain extends infinitely far in all directions outward from the ellipse, it cannot be contained within such a circle and is thus unbounded.

2 It is helpful to recall exactly what this notation means: y is constant, z is an independent variable, and x is a function of y and z. Begin, then, by differentiating the given equation for w bearing in mind that x is itself a function and thus applying the chain rule appropriately. This gives $\left(\frac{\partial w}{\partial z}\right)_y=\frac{\partial x}{\partial z}ye^{xy}-\frac{y}{z^2}-\cosh x\frac{\partial x}{\partial z}=\frac{\partial x}{\partial z}(ye^{xy}-\cosh x)-\frac{y}{z^2}$. To continue, we clearly need to find $\frac{\partial x}{\partial z}$. This can be done by implicitly differentiating the constraint equation with respect to z: $2xy\frac{\partial x}{\partial z}-\cos y+\frac{x+z(\partial x/\partial z)}{xz}=0$; solving this for $\frac{\partial x}{\partial z}$ gives $\frac{\partial x}{\partial z}=\frac{x(z\cos y-1)}{z(2x^2y+1)}$. We now substitute that into the previously-found equation for $\left(\frac{\partial w}{\partial z}\right)_y$: $\left(\frac{\partial w}{\partial z}\right)_y=\left(\frac{x(z\cos y-1)}{z(2x^2y+1)}\right)(ye^{xy}-\cosh x)-\frac{y}{z^2}$.

3a Consider the path $x=0$, which we analyze by simply plugging that into the limit expression: $\lim_{(x,y)\to(0,0)}\frac{0+y}{0-y}=\lim_{y\to 0}(-1)=-1$. Now consider the path $y=0$, again plugging that into the limit expression: $\lim_{x\to 0}\frac{x+0}{x-0}=\lim_{x\to 0}(1)=1$. Since we have found two different limits by taking different paths, the limit does not exist.

3b Multiply the limit expression by one in the following form: $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}\left(\dfrac{\sqrt{x^2+y^2+1}+1}{\sqrt{x^2+y^2+1}+1}\right)$

$= \lim\limits_{(x,y)\to(0,0)} \dfrac{\cancel{\left(x^2+y^2\right)}\left(\sqrt{x^2+y^2+1}+1\right)}{\cancel{x^2+y^2}} = \lim\limits_{(x,y)\to(0,0)}\left(\sqrt{x^2+y^2+1}+1\right)$. This can be evaluated simply by plugging in $(x,y)=(0,0)$, giving $\sqrt{0^2+0^2+1}+1=2$. Therefore $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}=2$.

4a We want to show that $\forall\varepsilon>0\ \exists\delta>0 : 0<\sqrt{(x+1)^2+(y-3)^2}<\delta \Rightarrow |3x-2y+9|<\varepsilon$. In classic delta-epsilon style, we begin with what we want and try to get what we want to start with: we rewrite $|3x-2y+0|<\varepsilon$ as $|3(x+1)-2(y-3)|<\varepsilon$, which we're permitted to call $3|x+1|+2|y-3|<\varepsilon$. Since $3+2=5$, we decide that $\forall\varepsilon>0$ we will let $\delta=\varepsilon/5$. Now for the actual proof:

$\sqrt{(x+1)^2+(y-3)^2}<\delta$ means that $\sqrt{(x+1)^2+(y-3)^2}<\varepsilon/5$. Multiplying both sides by 5 gives us the inequality $5\sqrt{(x+1)^2+(y-3)^2}<\varepsilon$, or $3\sqrt{(x+1)^2+(y-3)^2}+2\sqrt{(x+1)^2+(y-3)^2}<\varepsilon$. This can be written as $3|x+1|+2|y-3|<\varepsilon$, or as $|3(x+1)-2(y-3)|<\varepsilon$, and finally as $|3x-2y+9|<\varepsilon$ which is our goal. Therefore we have shown $\forall\varepsilon>0\ \exists\delta>0 : 0<\sqrt{(x+1)^2+(y-3)^2}<\delta \Rightarrow |3x-2y+9|<\varepsilon$, which by definition means $\lim\limits_{(x,y)\to(-1,3)}(3x-2y)=-0$.

4b We want to show that $\forall\varepsilon>0\ \exists\delta{>}0 : \sqrt{x^2+y^2}<\delta \Rightarrow \left|\dfrac{2x^3+3x^2y}{x^2+y^2}\right|<\varepsilon$. Starting again with the end, we can rewrite $\left|\dfrac{2x^3+3x^2y}{x^2+y^2}\right|<\varepsilon$ as $\left|\dfrac{2x^{\cancel{3}}+3\cancel{x^2}y}{\cancel{x^2}}\right|<\varepsilon \Rightarrow |2x+3y|<\varepsilon$ and that as $2|x|+3|y|<\varepsilon$. Again we find that we should let $\delta=\varepsilon/5$.

Given any $\varepsilon>0$, we will let $\delta=\varepsilon/5$. Then $\sqrt{x^2+y^2}<\delta$ means $\sqrt{x^2+y^2}<\varepsilon/5 \Rightarrow 5\sqrt{x^2+y^2}<\varepsilon$, and therefore $2\sqrt{x^2+y^2}+3\sqrt{x^2+y^2}<\varepsilon$, or $2\sqrt{x^2}+3\sqrt{y^2}<\varepsilon$ meaning $2|x|+3|y|<\varepsilon$, which can be combined to give $|2x+3y|<\varepsilon$, which we are permitted to write as $\left|\dfrac{2x^3+3x^2y}{x^2}\right|<\varepsilon$ and even as $\left|\dfrac{2x^3+3x^2y}{x^2+y^2}\right|<\varepsilon$. This is what we wanted; we have just shown that $\forall\varepsilon>0\ \exists\delta{>}0 : \sqrt{x^2+y^2}<\delta \Rightarrow \left|\dfrac{2x^3+3x^2y}{x^2+y^2}\right|<\varepsilon$, which by definition means $\lim\limits_{(x,y)\to(0,0)} \dfrac{2x^3+3x^2y}{x^2+y^2}=0$.

5a It is actually more convenient to do **5b** first, as the easiest way to find the curve of intersection of these two figures is to find each one's tangent plane at the given point and then find the line of intersection of the two tangent planes. We will thus return to this problem after completing **5b**.

5b Recall that given $z=f(x,y)$, the tangent plane at a point (x_0,y_0,z_0) is given by $z-z_0=\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}(x-x_0)$ $+\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}(y-y_0)$. With $z=2x^2+3y^2$, evaluating $\frac{\partial z}{\partial x}$ gives $\frac{\partial z}{\partial x}=4x$, which at $(-2,1,11)$ is -8; evaluating $\frac{\partial z}{\partial y}$ gives $\frac{\partial z}{\partial y}=6y$, which at $(-2,1,11)$ is 6. Therefore the tangent plane to the given surface at $(-2,1,11)$ is $z-11=-8(x+2)+6(y-1)$.

5a Now we have found one of the tangent planes; let its normal vector be $\mathbf{n}_1=\langle -8,6,-1\rangle$. Obviously the given plane is its own tangent plane at all points; it has normal vector $\mathbf{n}_2=\langle 0,1,0\rangle$. To find the line of intersection of these two planes, we take $\mathbf{n}_1\times\mathbf{n}_2=\langle 1,0,-8\rangle$. This is a direction vector for the line tangent to the curve of intersection of the two surfaces, and we already know that a point on it is $(-2,1,11)$, so we can determine that the line is given by $\langle x,y,z\rangle=(-2,1,11)+\langle 1,0,-8\rangle t$.

6a This is Case 1 of the chain rule, described on page 968 of Stewart's textbook, except that *y* is a function of both *r* and *t*. The only effect that this change has on the problem is that dy/dt must be written as $\partial y/\partial t$, so we have $\frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$.

6b This is rather trickier. Begin by distributing the $\partial/\partial t$ operator using the product rule: $\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}\right)=\frac{\partial^2 z}{\partial y\partial t}\cdot\frac{\partial y}{\partial t}+\frac{\partial^2 y}{\partial t^2}\cdot\frac{\partial z}{\partial y}$. Now let's analyze which terms need to be rewritten and which are in their final form: $\frac{\partial^2 z}{\partial y\partial t}$ and $\frac{\partial^2 y}{\partial t^2}$ both must be rewritten; the others are fine as-is. Since $\frac{\partial^2 z}{\partial y\partial t}$ is by definition $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial t}\right)$ and we have already found $\partial z/\partial t$, we can simply find $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}\right)=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}\right)+\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}\right)$ $=\frac{\partial^2 z}{\partial y\partial x}\cdot\frac{dx}{dt}+\frac{\partial^2 x}{\partial y\partial t}\cdot\frac{\partial z}{\partial x}+\frac{\partial^2 z}{\partial y^2}\cdot\frac{\partial y}{\partial t}$.

7a This is a matter of preference about how you like to take your samples: centered about the point in question or off to one side or the other? In this solution we will use the former method. Then we divide ΔI by ΔT over the smallest possible interval that is centered about the point at which $T=94°\text{F}$ and $H=65\%$, which is $I=114°\text{F}$. This gives $\left.\frac{\partial I}{\partial T}\right|_{\substack{T=94°\text{F}\\H=65\%}}\approx\frac{121°\text{F}-108°\text{F}}{96°\text{F}-92°\text{F}}=\frac{13°\text{F}}{4°\text{F}}=13/4$.

7b We will need our estimate for $\left.\frac{\partial I}{\partial T}\right|_{\substack{T=94°\text{F}\\H=65\%}}$ from part **a** and an estimate for $\left.\frac{\partial I}{\partial H}\right|_{\substack{T=94°\text{F}\\H=65\%}}$ at the same point for our tangent plane approximation. Just as we did in **7a**, we divide the smallest possible ΔI by the smallest possible ΔH centered at the same point: $\left.\frac{\partial I}{\partial H}\right|_{\substack{T=94°\text{F}\\H=65\%}}=\frac{118°\text{F}-111°\text{F}}{70\%-60\%}=\frac{7}{10}$ °F/%. Now the tangent plane approximation is given by $I-I_0=\left.\frac{\partial I}{\partial T}\right|_{\substack{T=94°\text{F}\\H=65\%}}(T-T_0)+\left.\frac{\partial I}{\partial H}\right|_{\substack{T=94°\text{F}\\H=65\%}}(H-H_0)$ or $I-114°\text{F}=\frac{13}{4}(T-94°\text{F})+\frac{7}{10}(H-65\%)$.

7c Plugging in these points gives $I - 114°\text{F} = \frac{13}{4}(95°\text{F} - 94°\text{F}) + \frac{7}{10}(63\% - 65\%)$, so $I - 114°\text{F} = \frac{13}{4}(1°\text{F}) + \frac{7}{10}(-2\%)$, meaning $I \approx 115.85°\text{F}$.

8 The formula with which to begin is $V = \ell wd$, in which V represents the volume of a right rectangular parallelepiped, ℓ its length, w its width, and d its depth. The first partial derivatives of V are $\partial V/\partial\ell = wd$, $\partial V/\partial w = \ell d$, and $\partial V/\partial d = \ell w$. Therefore $dV = wd \cdot dl + \ell d \cdot dw + \ell w \cdot dd$. (Admittedly, using d to represent both depth and differentials is terrible notation, but it remains standard.) Don't forget to convert feet into inches or inches into feet; in this case, we will do the latter: $dV = (3\text{ ft})(2\text{ ft})(1/12\text{ ft}) + (5\text{ ft})(2\text{ ft})(1/12\text{ ft}) + (5\text{ ft})(2\text{ ft})(1/24\text{ ft})$. Note that the first two differential factors are $1/12\text{ ft}$ because there is $1/2\text{ in} = 1/24\text{ ft}$ on each side, but the last one is $1/24\text{ ft}$ because the top is left open so there is $1/2\text{ in} = 1/24\text{ ft}$ on only one side. Evaluating this expression for dV gives $dV = 47/24\text{ ft}^3$.

Bonus Problem Note that 26, 28, and 17 are each one more than perfect square, cube, and fourth-power integer, respectively: 25, 27, and 16. So we define $f(x,y,z) = \sqrt{x}\sqrt[3]{y}\sqrt[4]{z}$. Now we create differentials: $\frac{\partial f}{\partial x} = \frac{\sqrt[3]{y}\sqrt[4]{z}}{2\sqrt{x}}$, $\frac{\partial f}{\partial y} = \frac{\sqrt{x}\sqrt[4]{z}}{3y^{2/3}}$, and $\frac{\partial f}{\partial z} = \frac{\sqrt{x}\sqrt[3]{y}}{4z^{3/4}}$. This means that we have the rather unwieldy $df = \frac{\sqrt[3]{y}\sqrt[4]{z}}{2\sqrt{x}}dx + \frac{\sqrt{x}\sqrt[4]{z}}{3y^{2/3}}dy + \frac{\sqrt{x}\sqrt[3]{y}}{4z^{3/4}}dz$. It becomes much more manageable when we put the near values we established for *x*, *y*, and *z*, along with the difference of 1 for *dx*, *dy*, and *dz*: $df = \frac{\sqrt[3]{27}\sqrt[4]{16}}{2\sqrt{25}}(1) + \frac{\sqrt{25}\sqrt[4]{16}}{3(27)^{2/3}}(1) + \frac{\sqrt{25}\sqrt[3]{27}}{4(16)^{3/4}}(1) = \frac{(3)(2)}{2(5)} + \frac{(5)(2)}{3(9)} + \frac{(5)(3)}{4(8)} = \frac{3}{5} + \frac{10}{27} + \frac{15}{32} = \frac{6217}{4230}$.