Multivariable Calculus
Review Problems — Chapter 15, part 2

Things to Know and Be Able to Do

- Everything from part one of this chapter
- Evaluate gradients and understand the meaning of this operation
- Use gradients to write tangent planes
- Evaluate and optimize directional derivatives
- Perform optimization for multivariate functions including the use of the method of Lagrange multipliers

Practice Problems

You may use a calculator. The original test, of course, required that you show relevant work.

1 Write an equation for the tangent plane to the surface $3x^2 + 2xy - y^2 = 15$ at the point $(2,3,4)$.

2 Find the derivative of $f(x,y,z) = x^3 + y^3 + 2xyz$ at $(2,-1,0)$ in the direction of $\langle 3,-2,1\rangle$.

3 Find parametric equations for the line tangent to the curve of intersection of the surfaces $z = x^2 - 3xy + y^2$ and $2x^2 + y^2 - 3z + 27 = 0$ at $(1,-2,11)$.

4 Show that $(0,-1/2)$ is a saddle point of the function $f(x,y) = 4xy^2 - 2x^2y - x$.

5 Consider the function $f(x,y) = xy$. Find the absolute maximum and absolute minimum values of this function on the half of the curve $x^2/9 + y^2/4 = 1$ where $y \geq 0$.

6 Consider the function $f(x,y,z) = xyz$ and the constraints $x + y + z = 4$ and $x - y - z = 3$.

6a Set up the problem of finding the function's extrema given the constraints with Lagrange multipliers. Show the complete system of equations that must be solved. Do not solve the system.

6b Represent the line of intersection of the two constraints parametrically and substitute this into the function to express the function in terms of one variable. Do not solve this equation.

6c Use the setup from either **a** or **b** to find the minimum value of the function given the constraints.

7 Consider $f(x,y,z) = axy^2 + byz + cx^3z^2$ for real constants *a*, *b*, and *c*. The function's maximum derivative at $(1,2,-1)$ is 64 and is in the direction of the positive *z*-axis. Find *a*, *b*, and *c*.

These questions were to be solved at home over a weekend with only a calculator and the Stewart textbook as resources:

8 Use Lagrange multipliers to find the minimum volume for a region bounded by the coordinate planes and a plane tangent to the surface $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ in the first octant.

9 Find all relative minima and maxima and saddle points of $f(x,y) = 2x^4 + y^3 - x - 12y + 20$. Justify the type of point for each.

10 Given $f(x,y) = 2x + 2y - x^2 - y^2$, find the absolute minimum and maximum values of the function on the square region bounded by the coordinate axes and the lines $x = 2$ and $y = 2$ in the first quadrant.

Answers

1 $18x-2y=30$ **2** $26/\sqrt{14}$

3 $\langle x,y,z\rangle=(1,-2,11)+\langle 17,20,-4\rangle t$

5 maximum: 3; minimum: -3

6c $7/32$ **7** $(a,b,c)=(6,24,-8)$ **8** $\frac{\sqrt{3}}{2}abc$

9 $(1/2,-2)$ is a saddle point; $(1/2,2)$ is a relative minimum

10 the absolute minimum is 2 at $(1,1)$; the absolute minimum is 0 at $(0,0)$, $(0,2)$, $(2,0)$, and $(2,2)$

Solutions

1 Let $f(x,y,z)=3x^2+2xy-y^2$; then $\frac{\partial f}{\partial x}=6x+2y$, $\frac{\partial f}{\partial y}=2x-2y$, and obviously $\frac{\partial f}{\partial z}=0$, so using equation (19) from page 984 of Stewart's textbook gives the tangent plane as $0=(6x+2y)\big|_{(2,3,4)}(x-2)+(2x-2y)\big|_{(2,3,4)}(y-3)$, or $0=18(x-2)-2(y-3)$. This may also be written as $18x-2y=30$.

2 Taking the gradient ∇f gives $\langle 3x^2+2yz,3y^2+2xz,2xy\rangle$. Evaluated at $(2,-1,0)$ this is $\nabla f(2,-1,0)=\langle 12,3,-4\rangle$. Normalizing the given direction vector (call it **a**) results in $\hat{\mathbf{a}}=\frac{\mathbf{a}}{\|\mathbf{a}\|}=\frac{\langle 3,-2,1\rangle}{\sqrt{14}}=\left\langle \frac{3}{\sqrt{14}},-\frac{2}{\sqrt{14}},\frac{1}{\sqrt{14}}\right\rangle$. The directional derivative for which we are looking results from dotting these vectors: $\langle 12,3,-4\rangle\cdot\left\langle \frac{3}{\sqrt{14}},-\frac{2}{\sqrt{14}},\frac{1}{\sqrt{14}}\right\rangle$ $=\frac{36}{\sqrt{14}}-\frac{6}{\sqrt{14}}-\frac{4}{\sqrt{14}}=\frac{26}{\sqrt{14}}$.

3 The strategy here is to find the tangent plane to each surface at this point and then find the planes' line of intersection. The first equation, letting $f_1(x,y)=x^2-3xy+y^2$ has $\frac{\partial f_1}{\partial x}=2x-3y$ and $\frac{\partial f_1}{\partial y}=2y-3x$. At $(1,-2,11)$, these evaluate to 8 and -7 respectively, so the tangent plane to the first surface is $z-11=8(x-1)-7(y+2)$; this can also be written as $8x-7y-z=11$. For the second equation, we will use the method we used in problem **1**: considering $f_2(x,y,z)=2x^2+y^2-3z+27$. Now $\frac{\partial f_2}{\partial x}=4x$, $\frac{\partial f_2}{\partial y}=2y$, and $\frac{\partial f_2}{\partial z}=-3$. Evaluating these at $(1,-2,11)$, gives 4, -4, and -3 respectively, so the tangent plane is $0=4(x-1)-4(y+2)-3(z-11)$, which might also be written as $4x-4y-3z=-21$. Now we have two planes with normal vectors $\mathbf{n}_1=\langle 8,-7,-1\rangle$ and $\mathbf{n}_2=\langle 4,-4,-3\rangle$ and we must find the planes' line of intersection. Taking the cross product of the normal vectors gives $\mathbf{n}_1\times\mathbf{n}_2=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 8 & -7 & -1\\ 4 & -4 & -3\end{vmatrix}=\langle 17,20,-4\rangle$ which is the direction vector of the line of intersection. We already know that the point of mutual tangency of the planes is a point on this line, so we can write parametric equations for the line as $\langle x,y,z\rangle=(1,-2,11)+\langle 17,20,-4\rangle t$.

4 We first need to show that $(0,-1/2)$ is a critical point; we find $\frac{\partial f}{\partial x}=4y^2-4xy-1$ which at $(0,-1/2)$ is 0; $\frac{\partial f}{\partial y}=8xy-2x^2$ which at $(0,-1/2)$ is also 0. Since both are 0, this is a critical point. To determine what type of critical point this is, we use the Second Derivatives Test. The function's three second derivatives are $\frac{\partial^2 f}{\partial x^2} = -4y$, $\frac{\partial^2 f}{\partial y \partial x} = 8y - 4x$, and $\frac{\partial^2 f}{\partial y^2} = 8x$. The function $D(x,y) = \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial y \partial x}\right)^2$ is $D(x,y) = (-4y)(8x) - (8y-4x)^2 = -16x^2 + 32xy - 64y^2$. Evaluating $D(0,-1/2)$ gives -16; according to the Second Derivatives Test, a negative result indicates a saddle point.

5 We use the method of Lagrange multipliers; we will let the constraint (the given ellipse) be $g(x,y) = x^2/9 + y^2/4$. Thus $\nabla f = \lambda \nabla g$ means $\begin{cases} \langle y,x \rangle = \lambda \langle \frac{2}{9}x, \frac{1}{2}y \rangle \\ x^2/9 + y^2/4 = 1 \end{cases}$. This system solves to (including positive y-values only) $(x,y) \in \left\{ \left(-\frac{3}{\sqrt{2}}, \sqrt{2}\right), \left(\frac{3}{\sqrt{2}}, \sqrt{2}\right) \right\}$. These evaluate to -3 and 3, respectively, but we must also check the end-points of the half of the curve. These are $(x,y) = (\pm 3, 0)$ which, when inputted to $f(x,y) = xy$, give 0. Therefore the absolute maximum value is 3 and the absolute minimum value is -3.

6a The setup for Lagrange multipliers with two constraints is $\nabla f = \lambda \nabla g + \mu \nabla h$; in this case, we have $f(x,y,z) = xyz$, $g(x,y,z) = x + y + z$, and $h(x,y,z) = x - y - z$. The gradients are $\nabla f = \langle yz, xz, xy \rangle$, $\nabla g = \langle 1,1,1 \rangle$, and $\nabla h = \langle 1,-1,-1 \rangle$. Therefore the system is as follows: $\begin{cases} \langle yz, xz, xy \rangle = \lambda \langle 1,1,1 \rangle + \mu \langle 1,-1,-1 \rangle \\ x + y + z = 4 \\ x - y - z = 3 \end{cases}$ which can be written less concisely, but probably more preferably to your instructor, as $\begin{cases} yz = \lambda + \mu \\ xz = \lambda - \mu \\ xy = \lambda - \mu \\ x + y + z = 4 \\ x - y - z = 3 \end{cases}$.

6b The two constraints represent planes with normal vectors $\langle 1,1,1 \rangle$ and $\langle 1,-1,-1 \rangle$; to find their line of intersection, we take the cross product of the normal vectors: $\langle 1,1,1 \rangle \times \langle 1,-1,-1 \rangle = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{vmatrix} = \langle 0,2,-2 \rangle$. This is a direction vector for the line of intersection. To find a point on the line let z be zero and then solve the system $\begin{cases} x + y + 0 = 4 \\ x - y - 0 = 3 \end{cases}$ to get $(x,y) = (7/2, 1/2)$ which, taken with $z = 0$, is a point on the line. Therefore the line may be represented parametrically as $\langle x,y,z \rangle = (7/2, 1/2, 0) + \langle 0,2,-2 \rangle t$; that is, $\begin{cases} x = 7/2 \\ y = 1/2 + 2t \\ z = -2t \end{cases}$. Substituting these values into the function gives $f(x,y,z) = xyz \to f(t) = \frac{7}{2}\left(\frac{1}{2} + t\right)(-2t) = -14t^2 - \frac{7}{2}t$.

6c If it weren't for the availability of CAS-enabled calculators, it would probably be easier to use the setup from part **b**, though since both have already been set up neither is very difficult; here we will demonstrate both methods.

6ci Using the setup from part **a**, the system we found can be solved by calculator to give $(x,y,z)=(7/2,1/4,1/4)$.

6cii Using the setup from part **b**, we need to find the extreme values of $f(t)$. This is done by differentiating $f(t)$ with respect to t and setting this equal to zero; $\frac{df}{dt}=-28t-\frac{7}{2}=0$ implies $t=-1/8$. Now plugging that t value into the parametric equation for the line of intersection gives $(x,y,z)=(7/2,1/4,1/4)$.

We have found the same point at which the value is at a minimum through both methods. Evaluating the function at that point gives $f(7/2,1/4,1/4)=(7/2)(1/4)(1/4)=7/32$.

7 The function's gradient is $\nabla f=\langle ay^2+3cx^2z^2,2axy+bz,by+2cx^3z\rangle$. At the given point, we have $\nabla f(1,2,-1)=\langle 4a+3c,4a-b,2b-2c\rangle$. The direction of its maximum derivative is parallel to the positive z-axis; to wit, $\langle 0,0,1\rangle$, meaning that $\langle 4a+3c,4a-b,2b-2c\rangle=\langle 0,0,1\rangle$. This gives $(a,b,c)=(3/32,3/8,-1/8)$. We did not, however, use the piece of information that the value of the directional derivative here is 64, which is suspicious, so we check that the values of a, b, and c we have found do indeed result in that value: they do not. We multiply a, b, and c by 64 each so that $(a,b,c)=(6,24,-8)$ and find that this results in the correct answer.

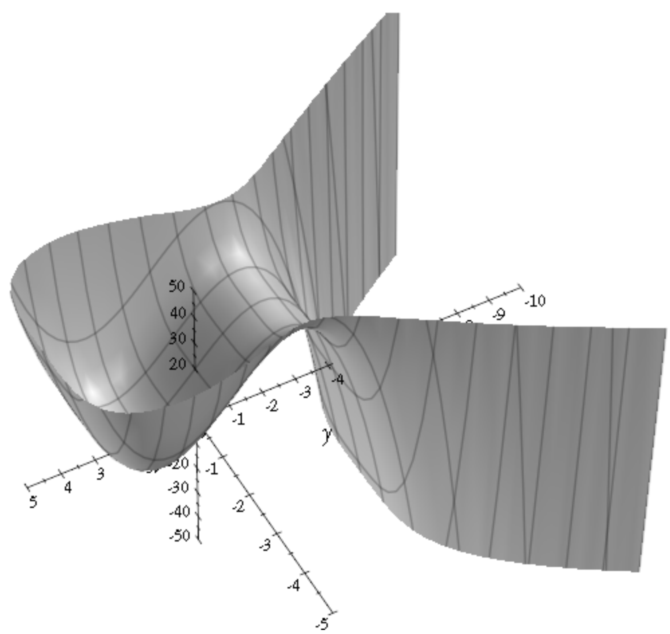
8 We must firstly find the tangent plane to the given surface (an ellipsoid) at some arbitrary point (x_0,y_0,z_0). To do this, consider $f(x,y,z)=x^2/a^2+y^2/b^2+z^2/c^2$ and take $\nabla f=\langle 2x/a^2,2y/b^2,2z/c^2\rangle$ which at (x_0,y_0,z_0) is $\nabla f(x_0,y_0,z_0)=\langle 2x_0/a^2,2y_0/b^2,2z_0/c^2\rangle$. According to equation (19) on page 984 of the Stewart textbook, we can use this to write the tangent plane as $\frac{2x_0}{a^2}(x-x_0)+\frac{2y_0}{b^2}(y-y_0)+\frac{2z_0}{c^2}(z-z_0)=0$, or equivalently in the form $2\left(\frac{x_0}{a^2}x+\frac{y_0}{b^2}y+\frac{z_0}{c^2}z\right)=2\left(\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}+\frac{z_0^2}{c^2}\right)$. Note that the second factor on the right side is 1 by the definition of the ellipsoid, since (x_0,y_0,z_0) is a point on it, and cancel a factor of 2 on each side. Now we know that $\frac{x_0}{a^2}x+\frac{y_0}{b^2}y+\frac{z_0}{c^2}z=1$ gives a tangent plane to the ellipsoid at (x_0,y_0,z_0).

The region in space described is a pyramid. To find its dimensions, we must work with the tangent plane we have found; it has x-, y-, and z-intercepts a^2/x_0, b^2/y_0, and c^2/z_0, respectively. The pyramid, then, has height c^2/z_0 and a right triangular base with legs a^2/x_0 and b^2/y_0. Its volume is given by $V=\frac{1}{3}Bh=\frac{1}{3}\left(\frac{1}{2}\left(\frac{a^2}{x_0}\right)\left(\frac{b^2}{y_0}\right)\right)\left(\frac{c^2}{z_0}\right)=\frac{a^2b^2c^2}{6x_0y_0z_0}$. This means we need to minimize $\frac{a^2b^2c^2}{6x_0y_0z_0}$ subject to the constraint $x_0^2/a^2+y_0^2/b^2+z_0^2/c^2=1$. Let $V(x_0,y_0,z_0)=\frac{a^2b^2c^2}{6x_0y_0z_0}$ and $C(x_0,y_0,z_0)=x_0^2/a^2+y_0^2/b^2+z_0^2/c^2$. We find the gradients for each of these functions: $\nabla V=\left\langle -\frac{a^2b^2c^2}{6x_0^2y_0z_0},-\frac{a^2b^2c^2}{6x_0y_0^2z_0},-\frac{a^2b^2c^2}{6x_0y_0z_0^2}\right\rangle$ and $\nabla C=\langle 2x_0/a^2,2y_0/b^2,2z_0/c^2\rangle$. The method of Lagrange multipliers requires that we solve the system $\begin{cases} \nabla V = \lambda \nabla C \\ C = 1 \end{cases}$ for (x_0, y_0, z_0). This system is the less concisely and more usefully written as $\begin{cases} -\dfrac{a^2b^2c^2}{6x_0^{\ 2}y_0z_0} = \dfrac{2\lambda x_0}{a^2} \\ -\dfrac{a^2b^2c^2}{6x_0y_0^{\ 2}z_0} = \dfrac{2\lambda y_0}{b^2} \\ -\dfrac{a^2b^2c^2}{6x_0y_0z_0^{\ 2}} = \dfrac{2\lambda z_0}{c^2} \\ x_0^{\ 2}/a^2 + y_0^{\ 2}/b^2 + z_0^{\ 2}/c^2 = 1 \end{cases}$ which we solve as follows:

Solving the first equation for z_0 gives $z_0 = -\dfrac{a^4b^2c^2}{12\lambda x_0^{\ 3}y_0}$; solving the second for z_0 gives $z_0 = -\dfrac{a^2b^4c^2}{12\lambda x_0 y_0^{\ 3}}$. We can set these equal to each other; $-\dfrac{a^4b^2c^2}{12\lambda x_0^{\ 3}y_0} = -\dfrac{a^2b^4c^2}{12\lambda x_0 y_0^{\ 3}}$ means $b^2x_0^{\ 2} = a^2y_0^{\ 2}$. Since we are concerned only with non-negative values of all the variables, we can take the square root of both sides and find $y_0 = \dfrac{b}{a}x_0$. Now solve the first and third equations for y_0 to get $y_0 = -\dfrac{a^4b^2c^2}{12\lambda x_0^{\ 3}z_0}$ and $y_0 = -\dfrac{a^2b^2c^4}{12\lambda x_0 z_0^{\ 3}}$ respectively; again we set them equal to each other. This eventually produces $z_0 = \dfrac{c}{a}x_0$. We now substitute these known values of y_0 and z_0 into the last (the constraint) equation: $\dfrac{x_0^{\ 2}}{a^2} + \dfrac{\left(\frac{b}{a}x_0\right)^2}{b^2} + \dfrac{\left(\frac{c}{a}x_0\right)^2}{c^2} = 1$, which is expanded to $\dfrac{x_0^{\ 2}}{a^2} + \dfrac{\cancel{b^2}}{a^2}\dfrac{x_0^{\ 2}}{\cancel{b^2}} + \dfrac{\cancel{c^2}}{a^2}\dfrac{x_0^{\ 2}}{\cancel{c^2}} = 1$, so $\dfrac{x_0^{\ 2}}{a^2} = \dfrac{1}{3}$; that is, $x_0 = \dfrac{a}{\sqrt{3}}$. Substituting this into previously-found equations gives $y_0 = \dfrac{b}{\sqrt{3}}$ and $z_0 = \dfrac{c}{\sqrt{3}}$.

Now that we have the minimum values for x_0, y_0, and z_0, we can find the minimum volume of the described region:

$$V = \frac{a^2b^2c^2}{6x_0y_0z_0} = \frac{a^2b^2c^2}{6\dfrac{a}{\sqrt{3}}\dfrac{b}{\sqrt{3}}\dfrac{c}{\sqrt{3}}} = \frac{\sqrt{3}}{2}abc.$$



9 Given $f(x,y) = 2x^4 + y^3 - x - 12y + 20$, we find the following significant partial derivatives: $\partial f/\partial x = 8x^3 - 1$, $\partial f/\partial y = 3y^2 - 12$, $\partial^2 f/\partial x^2 = 24x^2$, $\partial^2 f/\partial y^2 = 6y$, and $\dfrac{\partial^2 f}{\partial y \partial x} = 0$. Consider $D(x,y) = \left(\dfrac{\partial^2 f}{\partial x^2}\right)\left(\dfrac{\partial^2 f}{\partial y^2}\right) - \left(\dfrac{\partial^2 f}{\partial y \partial x}\right)^2 = 144x^2y$. Now we must solve the system $\begin{cases} \partial f/\partial x = 8x^3 - 1 \ = 0 \\ \partial f/\partial y = 3y^2 - 12 \ = 0 \end{cases}$ which gives $(x,y) = (1/2, \pm 2)$.

Plugging each of these points into $D(x,y)$ gives $D(1/2,2)=72$ and $D(1/2,-2)=-72$. Since the latter is negative, $(1/2,-2)$ is a saddle point. For the former, we consider $\left.\frac{\partial^2 f}{\partial x^2}\right|_{(1/2,2)}=6$. Since this is positive, $(1/2,2)$ is a relative minimum. In case you find yourself curious, a graph of this function is shown at right.

10 Given $f(x,y)=2x+2y-x^2-y^2$, we find $\partial f/\partial x=2-2x$ and $\partial f/\partial y=2-2y$. We must solve the system $\begin{cases}\partial f/\partial x=2-2x=0\\ \partial f/\partial y=2-2y=0\end{cases}$ which gives $(x,y)=(1,1)$. Since this is in the region given, we must consider it: $f(1,1)=2$.

Now we must consider the boundaries:

For $x=0$, we have $f(0,y)=2y-y^2$; then $\left.\frac{\partial f}{\partial y}\right|_{(0,y)}=2-2y$; setting that equal to zero gives $y=1$, so we must consider $f(0,1)=1$.

For $x=2$, we have $f(2,y)=2y-y^2$; then $\left.\frac{\partial f}{\partial y}\right|_{(2,y)}=2-2y$; setting that equal to zero gives $y=1$, so we must consider $f(2,1)=1$.

For $y=0$, we have $f(x,0)=2x-x^2$; then $\left.\frac{\partial f}{\partial x}\right|_{(x,0)}=2-2x$; setting that equal to zero gives $x=1$, so we must consider $f(1,0)=1$.

For $y=2$, we have $f(x,2)=2x-x^2$. Then $\left.\frac{\partial f}{\partial x}\right|_{(x,2)}=2-2x$; setting that equal to zero gives $x=1$, so we must consider $f(1,2)=1$.

Now we consider the four vertices of the square as follows: $f(0,0)=0$; $f(0,2)=0$; $f(2,0)=0$; and $f(2,2)=0$. This gives us a total of nine points to consider. Of these, the function's value is maximal at $(1,1)$ where it is 2; the functional is minimal at the four vertices, where it is 0.