Multivariable Calculus
Review Problems — Chapter 16

Things to Know and Be Able to Do

- Understand the Riemann sum definition of double and triple integrals and compare them and their applications with those of single integrals
- Understand and apply the formulas for average value and surface area with multiple integrals, including the derivation of these and their analogies to similar formulas for single integrals
- Use Fubini's Theorem for evaluating iterated integrals
- Change the order of integration when possible, and understand when it is not possible or not feasible to do so
- Evaluate double integrals in both Cartesian and polar coordinates, and triple integrals in Cartesian, cylindrical, and spherical coordinates, and use the Jacobian transformations for these conversions
- Use multiple integrals to find centers of mass and moments of inertia
- Apply the method of the Jacobian to evaluate integrals over complicated regions, including knowing when to use a transformation, finding a suitable transformation, and applying this transformation

Practice Problems

These problems should be done without a calculator. The original test, of course, required that you show relevant work.

1 Write an integral expression in Cartesian coordinates that represents the volume of the region cut from the cylinder $y^2+z^2=4$ by the planes $x=0$ and $x+z=3$. You need not evaluate the expression.

2 Evaluate the double integral $\int_0^1\int_{2y}^2 4\cos\left(x^2\right)dx\,dy$.

3 Evaluate the double integral $\int_0^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\frac{2}{\left(1+x^2+y^2\right)^2}dy\,dx$.

4 Rewrite the integral $\int_0^2\int_{-\sqrt{1-(y-1)^2}}^0 xy^2\,dxdy$ in polar coordinates. You need not evaluate it.

5 Set up, but do not evaluate, an integral expression that represents the *x*-coordinate of the center of mass of the region in the *xy*-plane bounded by the parabola $x=y-y^2$ and the line $x+y=0$ if the density δ of the region is given by $\delta(x,y)=x+y$.

6 A solid region in space is bounded above by the sphere $x^2+y^2+z^2=20$ and below by the paraboloid $z=x^2+y^2$. The density δ of the region is given by the function $\delta(x,y,z)=\frac{yz}{x}$. Write an integral expression using *cylindrical coordinates* that represents the mass of the region. You need not evaluate it.

7 Rewrite the integral expression $\int_{-1}^1\int_{y^2}^1\int_0^{1-x}dzdxdy$ using the order $\iiint dxdzdy$. You need not evaluate it.

8 Give a physical interpretation of the quantity $\int_0^{2\pi}\int_0^{\pi/6}\int_0^{8\cos\varphi}\rho^3\cos\varphi\sin\varphi\,d\rho\,d\varphi\,d\theta$. Include in your answer a description of the region of integration and the meaning of the integral. Be as specific as possible.

9 Use the transformation $u = y/x^2$, $v = x/y^2$ and the following steps to show that the area of the region in the first quadrant bounded by the curves $y = x^2$, $y = 2x^2$, $x = y^2$, and $x = 4y^2$ is 1/8. Hint: $y^3 = \dfrac{1}{uv^2}$ and $x^3 = \dfrac{1}{u^2 v}$.

9a Sketch the region bounded by the four given curves in the *xy*-plane.

9b Show that the Jacobian is $\dfrac{1}{3u^2v^2}$.

9c Express the area using an integral in the *uv*-plane, and show the evaluation to give $1/8$.

Bonus Problem Set up a triple integral expression using only *one* triple integral that represents the volume of the region that lies between the surfaces $z = x^2 + y^2$ and $z = 4x^2 + 4y^2$ between the surfaces $z = 1$ and $z = 4$. You need not evaluate it.

Answers

Your answers may vary, especially given that often several different orders of integration are possible. One way to check equivalence is have your calculator evaluate (or evaluate by hand) your answer and the given answer and determine whether the answers are equivalent; if they are, this is not conclusive, but it is a good sign.

1 $\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{0}^{3-z} dxdzdy$

2 $\sin 4$

3 $\pi/2$

4 $\int_{\pi/2}^{\pi}\int_{0}^{2\sin\theta} r^4\cos\theta\sin^2\theta drd\theta.$

5 $\bar{x} = \dfrac{\int_0^2\int_{-y}^{y-y^2} x(x+y)dxdy}{\int_0^2\int_{-y}^{y-y^2}(x+y)dxdy}$

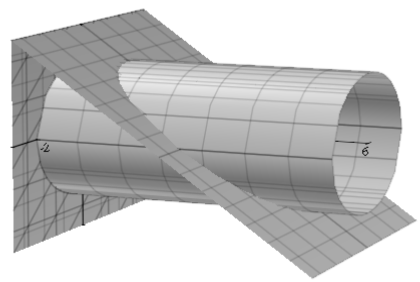
6 $m = \int_0^{2\pi}\int_0^2\int_{r^2}^{\sqrt{20-r^2}} rz\tan\theta dzdrd\theta$

7 $\int_{-1}^{1}\int_{0}^{1-y^2}\int_{y^2}^{1-z} dxdzdy$

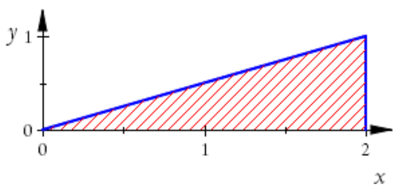
8 the moment about the xy-plane of a sphere with radius 4 centered about the x-axis and tangent to the yz-plane

Bonus Problem $\int_0^{2\pi}\int_1^4\int_{\sqrt{z}/2}^{\sqrt{z}} rdrdzd\theta$

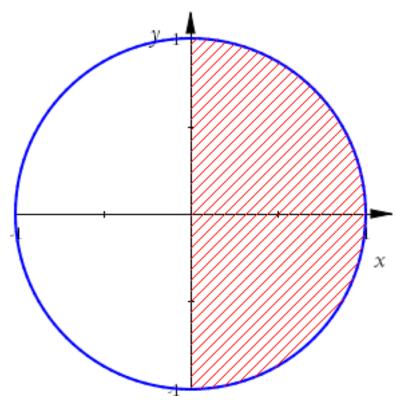
Solutions



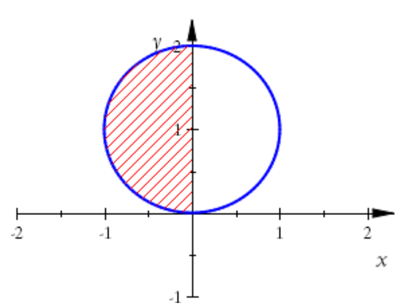
1 The region is shown at right. We see that x goes from 0 to $3-z$, z is governed by the cylinder so it goes from $-\sqrt{4-y^2}$ to $\sqrt{4-y^2}$, and x simply goes from -2 to 2. Since the integrand is 1, the integral is $\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{0}^{3-z} dxdzdy$.



2 The region is shown at left. We must switch the order of integration in order to perform the operation; the limts are rewritten so that the integral becomes $\int_0^2\int_0^{x/2} 4\cos(x^2)dydx$. Only now is it possible to evaluate this as $\int_0^2\int_0^{x/2} 4\cos(x^2)dydx = \int_0^2 4y\cos(x^2)\Big]_{y=0}^{y=x/2} dx = \int_0^2 2x\cos(x^2)dx$; perhaps you can antidifferentiate that by inspection. If not, use u-substitution with $u=x^2$ and $du=2xdx$ to write the integrand as $\cos u\,du$ of which an antiderivative is $\sin u = \sin(x^2)$, so we have found $\sin(x^2)\Big]_0^2 = \sin 4$.



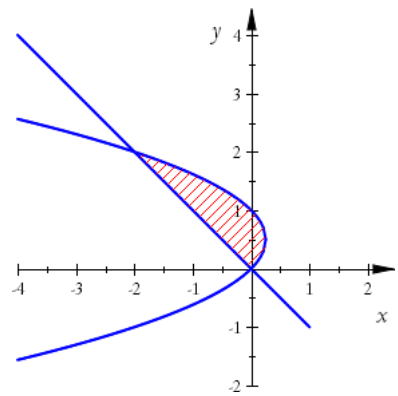
3 The region is shown at right. This will certainly be much easier to deal with in polar coordinates, in which the limits will be r going from 0 to 1 and θ going from $-\pi/2$ to $\pi/2$. Since in polar $x^2+y^2=r^2$, the integral (remembering to multiply by r for the Jacobian) is $\int_{-\pi/2}^{\pi/2}\int_0^1 \frac{2r}{(1+r^2)}drd\theta$. Letting $u=1+r^2$, $du=2rdr$ and we must antidifferentiate $\frac{du}{u^2}$ which might be $-1/u = -\frac{1}{1+r^2}$. Then we have $\int_{-\pi/2}^{\pi/2}\left(-\frac{1}{1+r^2}\right)\Big]_{r=0}^{r=1} d\theta = \int_{-\pi/2}^{\pi/2}(-1/2-(-1))d\theta = \theta/2\Big]_{-\pi/2}^{\pi/2} = \pi/2$.



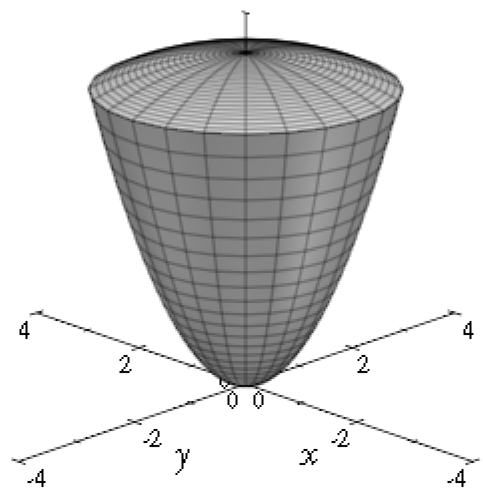
4 The region of integration is shown at right. The circle can be represented in polar coordinates by $r=2\sin\theta$ and the shaded part is given by $\theta\in[\pi/2,\pi]$. The integrand is transformed by $x=r\cos\theta$ and $y=r\sin\theta$: $xy^2 \to (r\cos\theta)(r\sin\theta)^2 = r^3\cos\theta\sin^2\theta$, which we multiply by r for the Jacobian to make the integral

$$\int_{\pi/2}^{\pi}\int_0^{2\sin\theta} r^4\cos\theta\sin^2\theta\,dr\,d\theta.$$

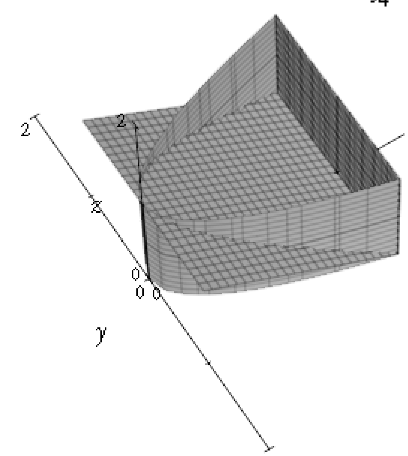
5 The region of integration is shown at right. Mass m is given by $m=\iint\delta(x,y)\,dA$, which in this case is $\int_0^2\int_{-y}^{y-y^2}(x+y)\,dx\,dy$. Finding the x-coordinate of the center of mass, $\overline{x}$, requires M_y, the region's moment about the y-axis. To find this, we multiply that very same integrand by x (the distance from the y-axis): $\int_0^2\int_{-y}^{y-y^2}x(x+y)\,dx\,dy$. Then $\overline{x}=\dfrac{M_y}{m}=\dfrac{\int_0^2\int_{-y}^{y-y^2}x(x+y)\,dx\,dy}{\int_0^2\int_{-y}^{y-y^2}(x+y)\,dx\,dy}$.



6 The region is shown at right. Mass m is given by $m=\iiint\delta(x,y,z)\,dV$, but since the question requires cylindrical coordinates we must rewrite $\delta(x,y,z)=\dfrac{yz}{x}$ as $\delta(r,\theta,z)=\dfrac{\cancel{r}\sin\theta z}{\cancel{r}\cos\theta}=z\tan\theta$. In cylindrical coordinates, the (top part of the) sphere on top is given by $z=\sqrt{20-r^2}$ and the paraboloid on the bottom is represented by $z=r^2$. Remember to multiply by r for the Jacobian and notice that the top and bottom boundaries of the region intersect in a circle of radius 2; the integral is then $\int_0^{2\pi}\int_0^2\int_{r^2}^{\sqrt{20-r^2}} rz\tan\theta\,dz\,dr\,d\theta$.



7 The region is shown at right. From the diagram, the integral can be written as $\int_{-1}^{1}\int_0^{1-y^2}\int_{y^2}^{1-z}dx\,dz\,dy$.



8 The figure $\rho=8\cos\varphi$ is a sphere of radius 4 $(=8/2)$ tangent to the yz-plane and centered about the x-axis; that is, it is centered at $(0,0,4)$. The integrand can be separated into $(\rho\cos\varphi)(\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta)$, which are z and dV, respectively. The z represents the distance from the xy-plane, so the integral represents the moment about the xy-plane of the aforedescribed sphere with uniform density.

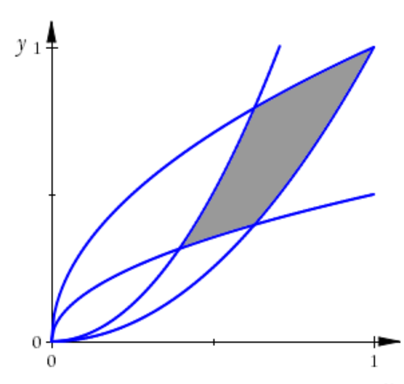
9a The region is shown at right.

9b Solving for x and y in terms of u and v gives $x=u^{-2/3}v^{-1/3}$ and $y=u^{-1/3}v^{-2/3}$. Then

$$J_2=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\partial x/\partial u & \partial x/\partial v\\ \partial y/\partial u & \partial y/\partial v\end{vmatrix}=\begin{vmatrix}-\tfrac{2}{3}u^{-5/3}v^{-1/3} & -\tfrac{1}{3}u^{-2/3}v^{-4/3}\\ -\tfrac{1}{3}u^{-4/3}v^{-2/3} & -\tfrac{2}{3}u^{-1/3}v^{-5/3}\end{vmatrix}$$

$$=\left(-\tfrac{2}{3}u^{-5/3}v^{-1/3}\right)\left(-\tfrac{2}{3}u^{-1/3}v^{-5/3}\right)-\left(-\tfrac{1}{3}u^{-4/3}v^{-2/3}\right)\left(-\tfrac{1}{3}u^{-2/3}v^{-4/3}\right)=\tfrac{4}{9}u^{-2}v^{-2}-\tfrac{1}{9}u^{-2}v^{-2}$$

$$=\frac{1}{3u^2v^2}.$$



9c The curves transform as follows:

$$y = x^2 \quad \to \quad u^{-1/3}v^{-2/3} = u^{-4/3}v^{-2/3} \quad \Rightarrow \quad u = 1$$
$$y = 2x^2 \quad \to \quad u^{-1/3}v^{-2/3} = 2u^{-4/3}v^{-2/3} \quad \Rightarrow \quad u = 2$$
$$x = y^2 \quad \to \quad u^{-2/3}v^{-1/3} = u^{-2/3}v^{-4/3} \quad \Rightarrow \quad v = 1$$
$$x = 4y^2 \quad \to \quad u^{-2/3}v^{-1/3} = 4u^{-2/3}v^{-4/3} \quad \Rightarrow \quad v = 4$$

The integrand, since we are simply finding the area of a region, is the (absolute value of the) Jacobian, with the limits we established above. We are therefore concerned with $\int_1^2\int_1^4 \frac{1}{3}u^{-2}v^{-2}dvdu = \int_1^2 -\frac{1}{3}u^{-2}v^{-1}\Big]_{v=1}^{v=4} du = \int_1^2 \frac{1}{4}u^{-2}du$ $= -\frac{1}{4}u\Big]_1^2 = 1/8$.

Bonus Problem A sketch of the region is shown at right. Parts of it have been cut away to make the region in question clear. These problem is best solved in cylindrical coordinates, in which the first two given bounding surfaces become $z = r^2$ and $z = 4r^2$. This means r will be going from $\sqrt{z}/2$ to $\sqrt{z}$. Clearly z goes from 1 to 4 and θ from 0 to 2π, so the integral (remembering to include a factor of r for the Jacobian) is $\int_0^{2\pi}\int_1^4\int_{\sqrt{z}/2}^{\sqrt{z}} r\,dr\,dz\,d\theta$.