Multivariable Calculus
Review Problems — Chapter 17, part 1

Things to Know and Be Able to Do

- Interpret vector fields in terms of fluid flow and in terms of their general definitions
- Understand the meaning of conservativism in the context of vector fields, and determine through any of several methods whether a field is conservative
- Understand the meaning of line integrals and use them to solve problems using parameterized paths
- Solve problems concerning work and circulation
- Use the Fundamental Theorem of Line Integrals when appropriate
- Apply Green's Theorem in its various forms
- Calculate divergence and curl without memorizing the explicit formula for curl
- Understand the intuitive interpretations of divergence, curl, work, and circulation
- Parameterize surfaces and find their areas using this technique

Practice Problems

You may use a calculator to work these problems. The original test, of course, required that you show relevant work.

1 Integrate the function $f(x,y,z)=x+\sqrt{y}-z^2$ over the path from $(0,0,0)$ to $(1,1,1)$ given by the curves $\mathbf{r}_1(t)=t\hat{\mathbf{i}}+t^2\hat{\mathbf{j}}$ for $0\le t\le 1$ and then $\mathbf{r}_2(t)=\hat{\mathbf{i}}+\hat{\mathbf{j}}+t\hat{\mathbf{k}}$ for $0\le t\le 1$.

2 Find the work done by the field $\mathbf{F}=xy\hat{\mathbf{i}}+y\hat{\mathbf{j}}-yz\hat{\mathbf{k}}$ over the curve $\mathbf{r}(t)=t\hat{\mathbf{i}}+t^2\hat{\mathbf{j}}+t\hat{\mathbf{k}}$ for $0\le t\le 1$ in the direction of increasing *t*.

3 Find the circulation along the curve $\mathbf{r}=-2\cos t\hat{\mathbf{i}}+2\sin t\hat{\mathbf{j}}+2t\hat{\mathbf{k}}$ for $0\le t\le \pi$ caused by the field $\mathbf{F}=-y\hat{\mathbf{i}}+x\hat{\mathbf{j}}+2\hat{\mathbf{k}}$.

4 Is the field $\mathbf{F}=ye^{x^2+y^2}\hat{\mathbf{i}}+xe^{x^2+y^2}\hat{\mathbf{j}}+z^3\tan^{-1}z\hat{\mathbf{k}}$ conservative? Justify your answer.

5 Evaluate the integral if possible; if not, explain why it is impossible.

5a $\int_{(1,1,1)}^{(1,2,3)} 3x^2dx+\frac{z^2}{y}dy+2z\ln y\,dz$

5b $\int_{(1,1,1)}^{(2,2,2)} \frac{dx}{y}+\left(\frac{1}{z}-\frac{x}{y^2}\right)dy-\frac{y}{z^2}dz$

6 Evaluate the line integral $\oint_C y^2dx+x^2dy$ where C is the triangle bounded by $x=0$, $x-y=0$, and $y=1$.

7 Use Green's Theorem to find the flux of **F** across C where $\mathbf{F}(x,y)=4xy^2\hat{\mathbf{i}}+4x^2y\hat{\mathbf{j}}$ and C is the circle $x^2+y^2=25$.

8 Use a surface integral to find the area of the surface cut from the paraboloid $x^2+y+z^2=1$ by the plane $y=0$.

9 Find the divergence and curl of the field $\mathbf{F}=\left(\sin^{-1}(xy)+2z\right)\hat{\mathbf{i}}+\left(\ln z+\cos y\right)\hat{\mathbf{j}}+z^2e^x\hat{\mathbf{k}}$.

Answers

1 $\dfrac{5^{3/2}+9}{6}$

2 1/2

3 0

4 No.

5a $9\ln 2$

5b 0

6 $-1/3$

7 1250π

8 $\dfrac{\pi}{6}\left(5^{3/2}-1\right)$

9 divergence: $\dfrac{y}{\sqrt{1-x^2y^2}}-\sin y+2ze^x$; curl: $-\dfrac{1}{z}\hat{\mathbf{i}}+\left(2-z^2e^x\right)\hat{\mathbf{j}}-\dfrac{x}{\sqrt{1-x^2y^2}}\hat{\mathbf{k}}$

Solutions

1 The first path is differentiated as $d\mathbf{r}_1=\left(\hat{\mathbf{i}}+2t\hat{\mathbf{j}}\right)dt$, so $\|d\mathbf{r}_1\|=\sqrt{1+4t^2}\,dt$. The second is differentiated as $d\mathbf{r}_2=\hat{\mathbf{k}}dt$, so $\|d\mathbf{r}_2\|=dt$. Integrating for the first path, substituting components of $\mathbf{r}_1$ for *x*, *y*, and *z* appropriately, gives $\int_0^1\left(t+|t|-0^2\right)\sqrt{1+4t^2}\,dt=\tfrac{1}{6}\left(4t^2+1\right)^{3/2}\Big]_0^1=\dfrac{5^{3/2}-1}{6}$. Integrating for the second path with the appropriate substitutions gives $\int_0^1\left(1+\sqrt{1-t^2}\right)dt=5/3$. These two components are added to give $\dfrac{5^{3/2}-1}{6}+\dfrac{5}{3}=\dfrac{5^{3/2}+9}{6}$.

2 Differentiating the path gives $d\mathbf{r}=\left(\hat{\mathbf{i}}+2t\hat{\mathbf{j}}+\hat{\mathbf{k}}\right)dt$. Writing **F** in terms of *t* gives $\mathbf{F}=t^3\hat{\mathbf{i}}+t^2\hat{\mathbf{j}}-t^3\hat{\mathbf{k}}$, so $W=\int_C\mathbf{F}\cdot d\mathbf{r}$ $=\int_0^1\left(t^3(1)+t^2(2t)-t^3(1)\right)dt=\int_0^1 2t^3dt=\tfrac{1}{2}t^4\Big]_0^1=1/2$.

3 Differentiating the path gives $d\mathbf{r}=\left(2\sin t\hat{\mathbf{i}}-2\cos t\hat{\mathbf{j}}+2\hat{\mathbf{k}}\right)dt$, and writing **F** in terms of *t* gives $\mathbf{F}=-2\sin t\hat{\mathbf{i}}-2\cos t\hat{\mathbf{j}}$ $+2\hat{\mathbf{k}}$. The integral is $\int_C\mathbf{F}\cdot d\mathbf{r}=\int_0^\pi\left((-2\sin t)(2\sin t)+(-2\cos t)(2\cos t)+(2)(2)\right)dt=\int_0^\pi\left(-4\sin^2t-4\cos^2t+4\right)dt$ $=\int_0^\pi(-4+4)dt=0$.

4 The easiest way to determine whether a field is conservative is to evaluate its curl.

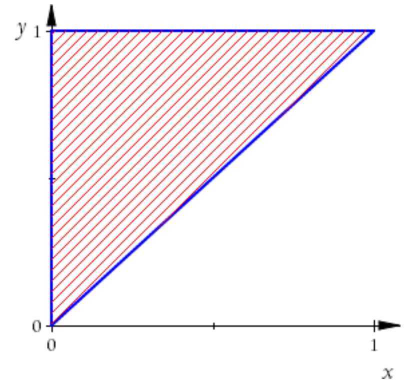
$$\nabla\times\mathbf{F}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ ye^{x^2+y^2} & xe^{x^2+y^2} & z^3\tan^{-1}z\end{vmatrix}=\left(\tfrac{\partial}{\partial y}\left(z^3\tan^{-1}z\right)-\tfrac{\partial}{\partial z}\left(xe^{x^2+y^2}\right)\right)\hat{\mathbf{i}}-\left(\tfrac{\partial}{\partial x}\left(z^3\tan^{-1}z\right)-\tfrac{\partial}{\partial z}\left(ye^{x^2+y^2}\right)\right)\hat{\mathbf{j}}+\left(\tfrac{\partial}{\partial x}\left(xe^{x^2+y^2}\right)-\tfrac{\partial}{\partial y}\left(ye^{x^2+y^2}\right)\right)\hat{\mathbf{k}}$$

The *x*- and *y*-components evaluate to 0; the *z*-component to $2e^{x^2+y^2}\left(x^2-y^2\right)$. This is not zero, so the curl is not the zero vector and the field is not conservative.

5a Consider that $\int_{(1,1,1)}^{(1,2,3)}3x^2dx+\dfrac{z^2}{y}dy+2z\ln y\,dz=\int_{(1,1,1)}^{(1,2,3)}\nabla f\cdot d\mathbf{s}$ in order to use the Fundamental Theorem of Line Integrals, so $f(x,y,z)=x^3+z^2\ln y+C$ for some constant C. Therefore we have $x^3+z^2\ln y\Big]_{(1,1,1)}^{(1,2,3)}=9\ln 2+1-1$ $=9\ln 2$.

5b The same procedure applies: in this case, $f(x,y,z)=x/y+y/z+C$ in order to have $\nabla f=\dfrac{1}{y}\hat{\mathbf{i}}+\left(1/z-x/y^2\right)\hat{\mathbf{j}}$ $-\dfrac{y}{z^2}\hat{\mathbf{k}}$. Then we evaluate $x/y+y/z\Big]_{(1,1,1)}^{(2,2,2)}=2-2=0$.

6 Recall Green's Theorem: $\oint_{\partial R} M dx + N dy = \iint_R \left(\partial N/\partial x \;\; - \partial M/\partial y\right) dA$ where R is a region and ∂R is its boundary with a counterclockwise orientation. In this case, $M = y^2$ and $N = x^2$, so $\dfrac{\partial N}{\partial x} = 2x$ and $\dfrac{\partial M}{\partial y} = 2y$. The region with which we are working is shown at right; Green's Theorem gives $\int_0^1 \int_x^1 (2x-2y)\,dy\,dx = \int_0^1 2xy - y^2 \Big]_{y=x}^{y=1} dx = \int_0^1 \left(-x^2+2x-1\right) dx = -\tfrac{1}{3}x^3 + x^2 - x \Big]_0^1$ $= -\tfrac{1}{3}$.



7 Another version of Green's Theorem states that $\oint_{\partial R} \mathbf{F}\cdot\mathbf{n}\,ds = \iint_R \nabla\cdot\mathbf{F}\,dA$. This is the relevant technique here. Evaluating $\nabla\cdot\mathbf{F}$ gives $\langle \partial/\partial x, \partial/\partial y\rangle \cdot \langle 4xy^2, 4x^2y\rangle = \frac{\partial}{\partial x}\left(4xy^2\right) + \frac{\partial}{\partial y}\left(4x^2y\right) = 4y^2 + 4x^2$, which since ∂R is a circle will be more convenient to evaluate in polar coordinates. Here $\nabla\cdot\mathbf{F} = 4x^2+4y^2$ becomes $\nabla\cdot\mathbf{F} = 4r^2$, which we multiply by r to get an integrand of $4r^3$. Putting in appropriate limits for the given circle gives $\int_0^{2\pi}\int_0^5 4r^3\,dr\,d\theta = \int_0^{2\pi} r^4\Big]_0^5 d\theta$ $= \int_0^{2\pi} 625\,d\theta = 1250\pi$.

8 We can parameterize the surface as $(x,y,z) = \left(u, 1-u^2-v^2, v\right)$. Then $\mathbf{r} = \left\langle u, 1-u^2-v^2, v\right\rangle$ gives $\dfrac{\partial\mathbf{r}}{\partial u} = \langle 1, -2u, 0\rangle$ and $\dfrac{\partial\mathbf{r}}{\partial v} = \langle 0, -2v, 1\rangle$ so $\left\|\dfrac{\partial\mathbf{r}}{\partial u}\times\dfrac{\partial\mathbf{r}}{\partial v}\right\| = \left\|\langle -2u, 1, -2v\rangle\right\| = \sqrt{4u^2+4v^2+1}$. We can use $r^2 = u^2+v^2$ to implement a rather different form of cylindrical coordinates; the Jacobian factor of r will remain, however, the same. Now $\left\|\dfrac{\partial\mathbf{r}}{\partial u}\times\dfrac{\partial\mathbf{r}}{\partial v}\right\| = \sqrt{4r^2+1}$. The relevant piece of the paraboloid has r going from 0 to 1 and θ going from 0 to 2π. Therefore we evaluate $\int_0^{2\pi}\int_0^1 r\sqrt{4r^2+1}\,dr\,d\theta$ with the substitution $w = 4r^2+1$ which gives $dw = 8r\,dr$. In order to find the necessary $\int r\sqrt{4r^2+1}\,dr$, we have $\int \frac{1}{8}\sqrt{w}\,dw = \frac{1}{12}w^{3/2} + C = \frac{1}{12}\left(4r^2+1\right) + C$ for some C. This brings us to evaluate $\frac{1}{12}\left(4r^2+1\right)^{3/2}\Big]_0^1 = \dfrac{5^{3/2}-1}{12}$. Thus we now must deal with evaluating $\int_0^{2\pi} \dfrac{5^{3/2}-1}{12} d\theta$ as $\dfrac{\pi}{6}\left(5^{3/2}-1\right)$.

9 The divergence of $\mathbf{F} = \left\langle F_x, F_y, F_z\right\rangle$, $\nabla\cdot\mathbf{F}$, is given by $\dfrac{\partial F_x}{\partial x} + \dfrac{\partial F_y}{\partial y} + \dfrac{\partial F_z}{\partial z}$. In this case, $F_x = \sin^{-1}(xy) + 2z$, $F_y = \ln z$ $+ \cos y$, and $F_z = z^2e^x$. Thus $\nabla\cdot\mathbf{F} = \frac{\partial}{\partial x}\left(\sin^{-1}(xy)+2z\right) + \frac{\partial}{\partial y}\left(\ln z + \cos y\right) + \frac{\partial}{\partial z}\left(z^2e^x\right) = \dfrac{y}{\sqrt{1-(xy)^2}} - \sin y + 2ze^x$.

Curl is quite a bit more involved: $\nabla\times\mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \sin^{-1}(xy)+2z & \ln z + \cos y & z^2e^x \end{vmatrix} = \Big\langle \frac{\partial}{\partial y}\left(z^2e^x\right) - \frac{\partial}{\partial z}\left(\ln z + \cos y\right),$

$-\left(\frac{\partial}{\partial x}\left(z^2e^x\right) - \frac{\partial}{\partial z}\left(\sin^{-1}(xy)+2z\right)\right), \frac{\partial}{\partial x}\left(\ln z + \cos y\right) - \frac{\partial}{\partial y}\left(\sin^{-1}(xy)+2z\right)\Big\rangle = \left\langle -1/z, 2 - z^2e^x, -\dfrac{x}{\sqrt{1-(xy)^2}}\right\rangle$.