Multivariable Calculus
Review Problems — Chapter 17, part 2

Things to Know and Be Able to Do

- Everything from part one of this chapter
- Apply surface integration to parameterized surfaces and understand the uses and meanings of this technique
- Understand the distinction between orientable and nonorientable surfaces, determine nonanalytically whether a surface is orientable or not, and understand why this is significant in the context of vector calculus
- Understand and use Stokes' Theorem and interpret it as a generalization of Green's Theorem
- Understand and use the Divergence Theorem (also known as Gauss's Theorem)

Practice Problems

You may use a calculator to work these problems. The original test, of course, required that you show relevant work.

1 A surface consists of the portion of the cone $z^2 = x^2 + y^2$ between the planes $z = 1$ and $z = 4$. The area density of any point on this surface is given by $\sigma(x, y, z) = 10 - z$ measured in grams per square meter. Use a surface integral to find the mass of the surface.

2 The velocity field of a fluid is given by $\mathbf{F}(x, y, z) = 2x\hat{\mathbf{i}} + 2y\hat{\mathbf{j}} + 3z\hat{\mathbf{k}}$.

2a Let *S* be the surface that is that portion of the paraboloid $z = 5 - x^2 - y^2$ inside the cylinder $x^2 + y^2 = 4$. Find the flux of **F** across *S* in the direction of the outward-pointing normal. (This is not a closed surface, so the Divergence Theorem does not apply. You must apply the definition of flux through a surface.)

2b Let *R* be the surface formed by the section of the plane $z = 1$ inside the cylinder $x^2 + y^2 = 4$. Find the flux of **F** across *R* in the direction of the downward-pointing normal. (Again, the Divergence Theorem does not apply.)

2c Now consider the closed surface that is the union of the two surfaces given in parts **a** and **b**; that is, the union of the paraboloid $z = 5 - x^2 - y^2$ inside the cylinder $x^2 + y^2 = 4$ along with its circular bottom. Use the Divergence Theorem to find the flux of **F** across this closed surface.

2d Describe how the answers to parts **a**, **b**, and **c** are related and why this is so.

3 Find the circulation counterclockwise around the triangle with vertices $(1,0,0)$, $(0,2,0)$, and $(0,0,3)$ that is in the vector field $\mathbf{F}(x, y, z) = -2xy\hat{\mathbf{i}} + x^2\hat{\mathbf{j}} + 4x\hat{\mathbf{k}}$.

4 Use a parameterization to find the flux due to the vector field $\mathbf{F}(x, y, z) = y\hat{\mathbf{i}} + z\hat{\mathbf{j}} - \hat{\mathbf{k}}$ outward (normally away from the *z*-axis) through the paraboloid $z = x^2 + y^2$ between the planes $z = 1$ and $z = 3$.

Answers

1 $108\sqrt{2}\pi$

2a 68π

2b -12π

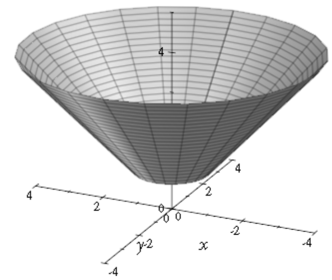
2c 56π

2d part **c** is the sum of parts **a** and **b**

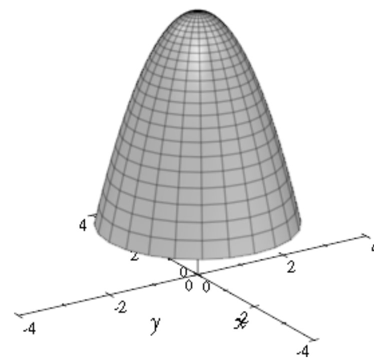
3 $-4/3$

4 -2π

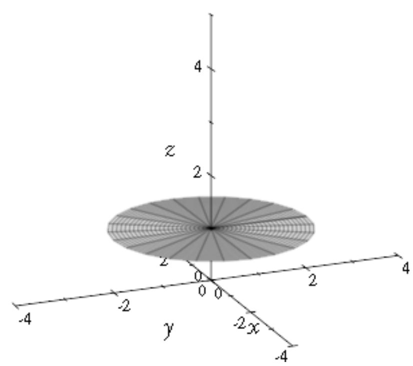
Solutions



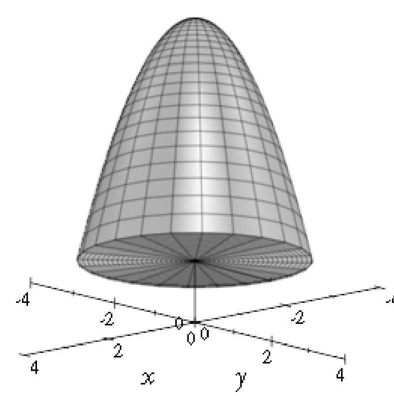
1 Parameterize the surface, shown at right, with $\mathbf{r}=\langle r\cos\theta, r\sin\theta, r\rangle$ for $r\in[1,4]$ and $\theta\in[0,2\pi)$. Then $\partial\mathbf{r}/\partial r=\langle\cos\theta,\sin\theta,1\rangle$ and $\partial\mathbf{r}/\partial\theta=\langle -r\sin\theta, r\cos\theta, 0\rangle$ meaning $\frac{\partial\mathbf{r}}{\partial r}\times\frac{\partial\mathbf{r}}{\partial\theta}=\langle -r\cos\theta,-r\sin\theta,r\rangle$ which gives $\left\|\frac{\partial\mathbf{r}}{\partial r}\times\frac{\partial\mathbf{r}}{\partial\theta}\right\|=|r|\sqrt{2}$. The density function ρ can be written as $\rho(x,y,z)=10-z\to\rho(r,\theta,z)=10-r$, and the mass of the surface *S* is given by evaluating $\iint_S\rho\,dS=\int_0^{2\pi}\int_1^4(10-r)\sqrt{2}|r|\,dr\,d\theta=108\sqrt{2}\pi$.



2a We begin by parameterizing the surface as $\mathbf{r}=\langle r\cos\theta, r\sin\theta, 5-r^2\rangle$ for $r\in[0,2]$ and $\theta\in[0,2\pi)$. This surface is shown at right. This gives us $\partial\mathbf{r}/\partial r=\langle\cos\theta,\sin\theta,-2r\rangle$ and $\partial\mathbf{r}/\partial\theta=\langle -r\sin\theta,r\cos\theta,0\rangle$ so $\frac{\partial\mathbf{r}}{\partial r}\times\frac{\partial\mathbf{r}}{\partial\theta}=\langle 2r^2\cos\theta, 2r^2\sin\theta, r\rangle$. The integral to find the flux is $\iint_S\mathbf{F}\cdot\left(\frac{\partial\mathbf{r}}{\partial r}\times\frac{\partial\mathbf{r}}{\partial\theta}\right)dS$, so we must write **F** in terms of *r* and θ: $\mathbf{F}=\langle 2r\cos\theta,2r\sin\theta,3(5-r^2)\rangle$. The integral thus becomes $\int_0^{2\pi}\int_0^2\langle 2r\cos\theta,2r\sin\theta,15-3r^2\rangle\cdot\langle 2r^2\cos\theta,2r^2\sin\theta,r\rangle\,dr\,d\theta=\int_0^{2\pi}\int_0^2(r^3+15r)\,dr\,d\theta=68\pi$.



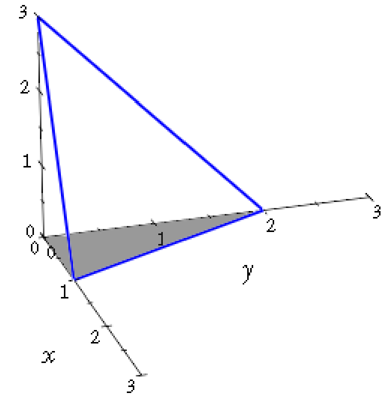
2b This (rather simple) surface is shown at right. It is parameterized as $\mathbf{r}=\langle r\cos\theta,r\sin\theta,1\rangle$ with $r\in[0,2]$ and $\theta\in[0,2\pi)$. Here $\partial\mathbf{r}/\partial r=\langle\cos\theta,\sin\theta,0\rangle$ and $\partial\mathbf{r}/\partial\theta=\langle -r\sin\theta,r\cos\theta,0\rangle$ so $\frac{\partial\mathbf{r}}{\partial r}\times\frac{\partial\mathbf{r}}{\partial\theta}=\langle 0,0,r\rangle$. Written in terms of *r* and θ, $\mathbf{F}=\langle 2r\cos\theta,2r\sin\theta,3\rangle$, so using the same type of integral gives $\int_0^{2\pi}\int_0^2\langle 0,0,r\rangle\cdot\langle 2r\cos\theta,2r\sin\theta,3\rangle\,dr\,d\theta=\int_0^{2\pi}\int_0^2 3r\,dr\,d\theta=12\pi$. However, the problem asks for the *downward* orientation; we have found the upward orientation, so we negate our answer to give -12π.



2c The union of the surfaces is shown at right. The divergence of **F**, $\nabla\cdot\mathbf{F}$, is calculated as follows: $\nabla\cdot\mathbf{F}=\langle\partial/\partial x,\partial/\partial y,\partial/\partial z\rangle\cdot\langle 2x,2y,3z\rangle=2+2+3=7$. We use cylindrical coordinates, in which $z\in[1,5-r^2]$, $r\in[0,2]$, and $\theta\in[0,2\pi)$. Remembering the extra factor of *r* due to the Jacobian, we evaluate the integral $\int_0^{2\pi}\int_0^2\int_1^{5-r^2}7r\,dz\,dr\,d\theta=56\pi$.

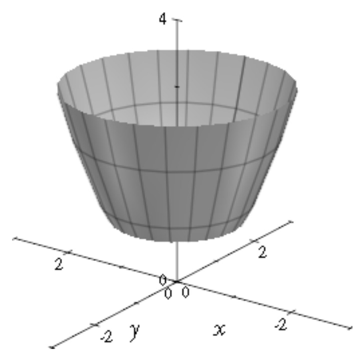
2d Naturally, the flux through a union of two surfaces would be the sum of the flux through each surface; indeed, generally, the flux through a union of surfaces is the sum of the flux through each of the surfaces. Therefore we have found, as expected that $56\pi=68\pi+(-12\pi)$. In other words, the answer to **c** is the sum of the answers to **a** and **b**.

3 The triangle is shown at right. The most convenient way to deal with this problem is with Stokes' Theorem, which states that $\int_{\partial S} \mathbf{F}\cdot d\mathbf{r} = \iint_S (\nabla\times\mathbf{F})\cdot d\mathbf{S}$.



We evaluate $\nabla\times\mathbf{F}$: $\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -2xy & x^2 & 4x \end{vmatrix} = \langle 0,-4,4x\rangle$. Now we must find a surface normal. Let the given points be A, B, and C respectively; we will be finding the plane that contains all three using the facts $\overrightarrow{AB} = \langle -1,2,0\rangle$ and $\overrightarrow{AC} = \langle -1,0,3\rangle$. Thus $\overrightarrow{AB}\times\overrightarrow{AC} = \langle 6,3,2\rangle$; this is a normal vector to the plane containing all three points. Let this be $\mathbf{n}$; then we have a unit vector $\hat{\mathbf{n}} = \dfrac{\langle 6,3,2\rangle}{\|\langle 6,3,2\rangle\|} = \dfrac{\langle 6,3,2\rangle}{7} = \langle 6/7, 3/7, 2/7\rangle$. The shaded region in the graph represents the area over which we must integrate $(\nabla\times\mathbf{F})\cdot d\mathbf{S}$; it is given by $y\in[0,2-2x]$ and $x\in[0,1]$, so the integral is $\int_0^1\int_0^{2-2x} \langle 0,-4,4x\rangle\cdot\langle 6/7,3/7,2/7\rangle\, dy\,dx = \int_0^1\int_0^{2-2x} \left(\tfrac{8}{7}x - \tfrac{12}{7}\right) dy\,dx$ $= -4/3$.

4 The surface is shown at right; a convenient parameterization is $\mathbf{r} = \langle r\cos\theta, r\sin\theta, r^2\rangle$. In this case, $r\in\left[1,\sqrt{3}\right]$ and $\theta\in[0,2\pi)$. Now we can find $\partial\mathbf{r}/\partial r = \langle \cos\theta, \sin\theta, 2r\rangle$ and $\partial\mathbf{r}/\partial\theta = \langle -r\sin\theta, r\cos\theta, 0\rangle$ which give $\dfrac{\partial\mathbf{r}}{\partial x}\times\dfrac{\partial\mathbf{r}}{\partial y} = \langle -2r^2\cos\theta, -2r^2\sin\theta, r\rangle$. Writing $\mathbf{F}$ in terms of our parameters results in $\mathbf{F} = \langle r\sin\theta, r^2, -1\rangle$. Finding flux in terms of a parameterized surface takes the form $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}\cdot\left(\dfrac{\partial\mathbf{r}}{\partial r}\times\dfrac{\partial\mathbf{r}}{\partial\theta}\right)dA$,



which here is $\int_0^{2\pi}\int_1^{\sqrt{3}} \langle r\sin\theta, r^2, -1\rangle\cdot\langle -2r^2\cos\theta, -2r^2\sin\theta, r\rangle\, dr\,d\theta = \int_0^{2\pi}\int_1^{\sqrt{3}} \left(-2r^3\sin\theta\cos\theta - 2r^4\cos\theta - r\right) dr\,d\theta$ $= -2\pi$.