AP Calculus BC
Review — Limits and Delta-Epsilon

Formal Definitions of Limits

It would be foolish to memorize each of these. Know the first one and be familiar enough with the others that you can use intelligent reasoning to figure them out. Actually, the list isn't even complete; forms such as $\lim\limits_{x\to a^{\pm}} f(x) = \mp\infty$ *have been omitted.*

$\lim\limits_{x\to a} f(x) = L$ means that $\forall \varepsilon > 0 \exists \delta > 0 : 0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

"for every positive ε, there exists a positive δ such that if x is within δ of a, then $f(x)$ is within ε of the limit."

$\lim\limits_{x\to a^+} f(x) = L$ means that $\forall \varepsilon > 0 \exists \delta > 0 : a < x < a + \delta \Rightarrow |f(x) - L| < \varepsilon$

$\lim\limits_{x\to a^-} f(x) = L$ means that $\forall \varepsilon > 0 \exists \delta > 0 : a - \delta < x < a \Rightarrow |f(x) - L| < \varepsilon$

$\lim\limits_{x\to \infty} f(x) = L$ (a horizontal asymptote) means that $\forall \varepsilon > 0 \exists M > 0 : x > M \Rightarrow |f(x) - L| < \varepsilon$

"for every positive ε, there exists a positive M such that if x is greater than M, then $f(x)$ is within ε of the limit."

$\lim\limits_{x\to -\infty} f(x) = L$ (a horizontal asymptote) means that $\forall \varepsilon > 0 \exists M < 0 : x < M \Rightarrow |f(x) - L| < \varepsilon$

$\lim\limits_{x\to a} f(x) = \infty$ (a vertical asymptote) means that $\forall N > 0 \exists \delta > 0 : 0 < |x-a| < \delta \Rightarrow f(x) > N$

$\lim\limits_{x\to a} f(x) = -\infty$ (a vertical asymptote) means that $\forall N < 0 \exists \delta > 0 : 0 < |x-a| < \delta \Rightarrow f(x) < N$

$\lim\limits_{x\to \infty} f(x) = \infty$ means that $\forall N > 0 \exists M > 0 : x > M \Rightarrow f(x) > N$

"for every positive N, there exists a positive M such that if x is greater than M, then $f(x)$ is greater than N."

$\lim\limits_{x\to -\infty} f(x) = \infty$ means that $\forall N > 0 \exists M < 0 : x < M \Rightarrow f(x) > N$

$\lim\limits_{x\to \infty} f(x) = -\infty$ means that $\forall N < 0 \exists M > 0 : x > M \Rightarrow f(x) > N$

$\lim\limits_{x\to -\infty} f(x) = -\infty$ means that $\forall N < 0 \exists M < 0 : x < M \Rightarrow f(x) < N$

Basically, if a is finite, δ is the tolerance around it; if $a \to \pm\infty$, we use M. If L is finite, its tolerance is ε ; if $f(x) \to \pm\infty$, we use N.

Following are some sample proofs. Make sure you understand what's going on. Don't just regurgitate the steps.

Proofs: Limits of Linear Functions

Suppose we want to prove that $\lim_{x\to 3}(2x+1)=7$. That is, we want to show that $\forall\varepsilon>0\exists\delta>0$: $0<|x-3|<\delta \Rightarrow |(2x+1)-7|<\varepsilon$.

We begin with the scratch work: starting with the part involving ε and trying to turn that into the part involving δ, as follows—

$$|(2x+1)-7|<\varepsilon$$
$$|2x-6|<\varepsilon$$
$$|2(x-3)|<\varepsilon$$
$$2|x-3|<\varepsilon$$
$$|x-3|<\tfrac{1}{2}\varepsilon$$

Since we now have something involving $|x-3|$, we've accomplished our goal. Now begins the proof. Having just shown that $|x-3|<\frac{1}{2}\varepsilon$ and planning to assume $|x-3|<\delta$, we state

$\forall\varepsilon>0$, let $\delta=\frac{1}{2}\varepsilon$ ("for every positive ε, let δ equal half of ε.")

The proof starts with the assumption: $|x-3|<\delta$. The steps are as follows:

$$\left.\begin{array}{l} |x-3|<\delta \\ |x-3|<\frac{1}{2}\varepsilon \end{array}\right\}\text{substituting our chosen value for } \delta$$

$$\left.\begin{array}{l} 2|x-3|<\varepsilon \\ |2(x-3)|<\varepsilon \\ |2x-6|<\varepsilon \\ |(2x+1)-7|<\varepsilon \end{array}\right\}\text{algebra to get something that looks like } |f(x)-L|<\varepsilon$$

W[5]; thus, we have shown that $\forall\varepsilon>0\exists\delta=\frac{1}{2}\varepsilon: 0<|x-3|<\delta\Rightarrow|(2x+1)-7|<\varepsilon$

meaning that $\lim_{x\to 3}(2x+1)=7$.

Proofs: Limits of Nonlinear Functions

Suppose we want to prove that $\lim_{x\to 4}\left(x^2-2x-3\right)=5$. That is, we want to show that $\forall\varepsilon>0\exists\delta>0$: $0<|x-4|<\delta\Rightarrow\left|\left(x^2-2x-3\right)-5\right|<\varepsilon$.

We begin with the scratch work: starting with the part involving ε and trying to turn that into the part involving δ, as follows—

$$\left|\left(x^2-2x-3\right)-5\right|<\varepsilon\Rightarrow\left|x^2-2x-8\right|<\varepsilon\Rightarrow\left|(x-4)(x+2)\right|<\varepsilon$$

We can see there's an $x-4$, which we want, but we need to get rid of the $x+2$. Remember that we're interested in very small values of δ, so we can restrict its value. Perhaps we could say $\delta<1$. Since we'll be assuming $|x-4|<\delta$ and thus $|x-4|<1$, we know $-1<x-4<1$, or if we add six to all three sides, $5<x+2<7$. Now we know $x+2>7$, so we can proceed.

$\left|(x-4)(x+2)\right|<\varepsilon$

And if $x+2<7$, then $\left|(x-4)(7)\right|<\left|(x-4)(x+2)\right|$,

so $\left|7(x-4)\right|<\varepsilon$, or $|x-4|<\frac{1}{7}\varepsilon$.

Since we now have something involving $|x-4|$, we've accomplished our goal. Now begins the proof. We can't just state "let $\delta=\frac{1}{7}\varepsilon$", though, because we also have restricted $\delta<1$. Therefore, we instead say

$\forall\varepsilon>0$, let $\delta=\min\left\{1,\frac{1}{7}\varepsilon\right\}$ ("one or one-seventh of ε, whichever's less.")

The proof starts with the second possible value for δ:

$|x-4|<\delta\Rightarrow|x-4|<\frac{1}{7}\varepsilon$

$7|x-4|<\varepsilon\Rightarrow\left|7(x-4)\right|<\varepsilon$

But now we need to deal with the 7. Since from the "let" statement we know $\delta<1$, we work backwards from that:

$|x-4|<\delta\Rightarrow|x-4|<1$

$-1<x-4<1$, and now we want to get something involving 7, so we add 6 to all three sides:

$5<x+2<7$

so we can substitute $x+2$ for the 7:

$\left|(x+2)(x-4)\right|<\left|7(x-4)\right|<\varepsilon$

$\left|x^2-2x-8\right|<\varepsilon\Rightarrow\left|\left(x^2-2x-3\right)-5\right|<\varepsilon$

W^5; thus, we have shown that

$\forall\varepsilon>0\exists\delta=\min\left\{1,\frac{1}{7}\varepsilon\right\}:0<|x-4|<\delta\Rightarrow\left|\left(x^2-2x-3\right)-5\right|<\varepsilon$

meaning that $\lim_{x\to 4}\left(x^2-2x-3\right)=5$.

Proofs: Infinite Limits

Suppose we want to prove that $\lim_{x\to -3}\frac{1}{(x+3)^4}=\infty$. That is, we want to show that $\forall N>0\exists\delta>0:$ $0<|x+3|<\delta\Rightarrow\frac{1}{(x+3)^4}>N$.

We begin with the scratch work: starting with the part involving N and trying to solve it for $x+3$:

$$\frac{1}{(x+3)^4}>N$$

$$1>N(x+3)^4 \quad \left(\text{note that doing this requires } (x+3)^4>0\text{, which is always true}\right)$$

$$\frac{1}{N}>(x+3)^4$$

$$\sqrt[4]{\frac{1}{N}}>x+3$$

$$\frac{1}{\sqrt[4]{N}}>x+3$$

Since we now have found an upper bound on $x+3$, we've accomplished our goal. Now begins the proof. We say

$$\forall N>0\text{, let } \delta=\frac{1}{\sqrt[4]{N}}$$

The proof starts with the assumption: $|x+3|<\delta$. The steps are as follows:

$$|x+3|<\delta$$

$$|x+3|<\frac{1}{\sqrt[4]{N}}$$

$$|x+3|^4=(x+3)^4<\frac{1}{N}$$

$$N(x+3)^4<1$$

$$N<\frac{1}{(x+3)^4}\Rightarrow\frac{1}{(x+3)^4}>N$$

W[5]; thus, we have shown that

$$\forall N>0\exists\delta=\frac{1}{\sqrt[4]{N}}:0<|x+4|<\delta\Rightarrow\frac{1}{(x+3)^4}>N\text{, meaning that } \lim_{x\to -4}\frac{1}{(x+3)^4}=\infty.$$

Proofs: Limits at Infinity

Suppose we want to prove that $\lim_{x\to\infty}\left(\frac{2}{x}+3\right)=3$. That is, we want to show that $\forall\varepsilon>0\exists M>0:$ $x>M\Rightarrow\left|\left(\frac{2}{x}+3\right)-3\right|<\varepsilon$.

We begin with the scratch work: starting with the part involving ε and trying to solve it for x:

$$\left|\left(\frac{2}{x}+3\right)-3\right|<\varepsilon$$

$$\left|\frac{2}{x}\right|<\varepsilon$$

Now we'd like to multiply by x, but then we need to make sure $x>0$. Since we're interested in $x\to\infty$, that's reasonable. Let $x>0$; then, we can drop the absolute value.

$$\frac{2}{x}<\varepsilon$$

$$2<\varepsilon x$$

$$\frac{2}{\varepsilon}<x$$

Since we now have found a lower bound on x, we've accomplished our goal. Now begins the proof. We say $\forall\varepsilon>0$, let $M=\frac{2}{\varepsilon}$.

The proof starts with the assumption: $x>M$. The steps are as follows:

$$x>M\Rightarrow x>\frac{2}{\varepsilon}$$

$$\varepsilon x>2$$

$$\varepsilon>\frac{2}{x}$$

Absolute value signs are free, so let's throw them in: $\left|\frac{2}{x}\right|<\varepsilon$

So is adding and subtracting: $\left|\left(\frac{2}{x}+3\right)-3\right|<\varepsilon$

W^5; thus, we have shown that

$\forall\varepsilon>0\exists M=\frac{2}{\varepsilon}:x>M\Rightarrow\left|\left(\frac{2}{x}+3\right)-3\right|<\varepsilon$, meaning that $\lim_{x\to\infty}\left(\frac{2}{x}+3\right)=3$.

Continuity

Recall the **definition of continuity**: a function $f(x)$ is continuous at $x=a$ if and only if $f(a)=\lim_{x\to a} f(x)$. Note that this necessarily implies the following:

- $\lim_{x\to a} f(x)$ exists
- $\lim_{x\to a^+} f(x)=\lim_{x\to a^-} f(x)$
- $f(a)$ exists

Then we can say a function is continuous on an open interval (a,b) iff $\lim_{x\to c} f(x)=f(c)$ for every $c:c\in(a,b)$. The definition for continuity on a closed $[a,b]$ is analogous.

Examples of functions that are continuous on their domain include polynomial functions, the sine and cosine functions, the absolute value function, exponential functions, and logarithmic functions.

Examples of functions with discontinuities are the floor and ceiling functions (at every integer), the tangent function (at every integer multiple of $\frac{\pi}{2}$) and the other nonelementary trigonometric functions, the signum function $\left(\operatorname{sgn} x=\frac{|x|}{x}\right)$ at $x=0$, and the canonical example of $\frac{\sin x}{x}$ at $x=0$. Many rational functions also have discontinuities.

A **discontinuity** at $x=a$ is called **removable** if $\lim_{x\to a} f(x)$ exists; that is, if we can "fill in the hole". For example, $\frac{\sin x}{x}$ has a removable discontinuity at $x=0$, but the discontinuities of $\tan x$ are nonremovable.

The **Intermediate Value Theorem** (IVT) states that if $f(x)$ is continuous on $[a,b]$, then $f(x)$ assumes every value c for $a<b<c$. That is, continuous functions don't jump around; they go everywhere between two given points.

Secant and Tangent Lines

One of the two central problems of calculus is finding the **slope of a tangent line**. We use the **secant line approximation** as a way to solve this: a secant to $f(x)$ going from $x=a$ to $x=b$ has slope $m=\dfrac{f(a)-f(b)}{a-b}$. If we let $b\to a$, then we have the slope of the tangent line at $x=a$: $m=\lim_{b\to a}\dfrac{f(a)-f(b)}{a-b}$, which can be evaluated—admittedly, it's often tedious, but a large part of this course is learning how to evaluate that limit by alternative means.

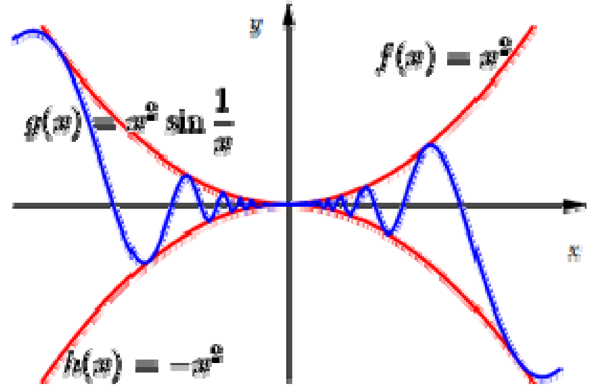
Thus we *define* that the tangent line to the function $y=f(x)$ at $(a,f(a))$ has slope $m=\lim_{x\to a}\dfrac{f(x)-f(a)}{x-a}$. It's then easy to write an equation for the tangent line using point-slope form; that is, $y-y_0=m(x-x_0)$ represents the line with slope m passing through (x_0,y_0).

If we define $h=x-a$, then $x=a+h$, and we can write an equivalent limit for the slope of the tangent line: $m=\lim_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$. This is sometimes easier to evaluate, and sometimes more difficult.

Recall from physics (or learn it now if you haven't taken physics) that the **slope of a position function** gives the velocity at that point.

The Squeeze Theorem

The **Squeeze Theorem** states that if a function $g(x)$ is bounded between two other functions $f(x)$ and $h(x)$ near $x = a$, and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x)$, then $\lim_{x \to a} g(x)$ is the limit of the other two. See the picture for a demonstration.



Symbols and Notation

$\exists$ means "there exists"
$:$ and $|$ mean "such that"
$\forall$ means "for every"
$\to$ means "approaches"
You may find it convenient to read $|x - a| < \delta$ as "x is within δ of a", and so forth.