AP Calculus BC
Review — Derivatives, Part I

Things to Know

- The derivative of a function at a point may be interpreted as the slope of a tangent line to the graph at that point.
- The derivative of a function is itself a function.
- The derivative of a function $f(x)$ with respect to x is denoted as $\frac{df}{dx}$ or $f'(x)$.
- The derivative of a function $f(x)$ at $x=a$ is denoted as $\left.\frac{df}{dx}\right|_{x=a}$ or $f'(a)$.
- A derivative of a function $f(x)$ with respect to x at a point $x=a$ is defined as $f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ or $f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$; these definitions are equivalent given the definition $h=x-a$.
- Be able to evaluate limits by recognizing them as derivatives of either of the forms above
- Not all functions are differentiable; for example, $f(x)=|x|$ is nondifferentiable at $x=0$ because it is "sharp" at that point. Continuity is necessary but not sufficient for differentiability.
- The Sum Rule: $\frac{d}{dx}\left(f(x)+g(x)\right)=\frac{df}{dx}+\frac{dg}{dx}$
- The Constant Multiple Rule: $\frac{d}{dx}\left(cf(x)\right)=c\frac{df}{dx}$
- The Sum Rule may be combined with the Constant Multiple Rule to form the Difference Rule: $\frac{d}{dx}\left(f(x)-g(x)\right)=\frac{df}{dx}-\frac{dg}{dx}$
- The Product Rule: $\frac{d}{dx}\left(f(x)g(x)\right)=f(x)\frac{dg}{dx}+g(x)\frac{df}{dx}$
- The Quotient Rule: $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{g(x)\frac{df}{dx}-f(x)\frac{dg}{dx}}{g(x)^2}$ (convenient mnemonic: "low dee-high minus high dee-low/draw a line and square below")
- The Power Rule: $\frac{d}{dx}\left(x^n\right)=nx^{n-1}$ and its derivation for positive integers n with the Binomial Theorem (you needn't be able to do it, but you should understand it). Know that this is valid for all real n even though we have only proven it for the integers.
- Important limits: $\lim_{x\to 0}\frac{\sin x}{x}=1$ and $\lim_{x\to 0}\frac{\cos x-1}{x}=0$

- The derivatives of trigonometric functions (you may find it easier to memorize only the first three and derive the latter three when necessary)
  - $\frac{d}{dx}(\sin x) = \cos x$
  - $\frac{d}{dx}(\cos x) = -\sin x$
  - $\frac{d}{dx}(\tan x) = \sec^2 x$
  - $\frac{d}{dx}(\sec x) = \tan x \sec x$
  - $\frac{d}{dx}(\csc x) = -\cot x \csc x$
  - $\frac{d}{dx}(\cot x) = -\csc^2 x$
- Find elementary antiderivatives by recognizing functions as derivatives using any of the following rules or combinations thereof: Power Rule, Product Rule, Quotient Rule. Remember to add a constant.

Practice Problems

Do not use a calculator.

1 Find each derivative.

a Find $f'(x)$ if $f(x) = 4\cos x - \dfrac{5}{x^3} + \pi$

b Find $\dfrac{dp}{dx}$ if $p = \dfrac{\cos x}{1+x^2}$

c Find $h'(x)$ if $h(x) = 2x^{\sqrt{3}} + \csc x + \dfrac{x^4}{6}$

d Find $\dfrac{dy}{dx}$ if $y = \tan x \csc x$

e Find $\dfrac{dx}{dt}$ if $x = \dfrac{-9}{\sqrt[4]{t^5}} + \sec t - \dfrac{\sqrt{5}}{\pi}$

2 Suppose that y and z are differentiable functions of x, and suppose that some values for $y(x)$, $z(x)$, $y'(x)$, and $z'(x)$ are given in the table to the right. Find the values of the following derivatives:

a $\dfrac{d}{dx}(yz)$ at $x = 2$

b $\dfrac{d}{dx}\left(\dfrac{z}{y}\right)$ at $x = 4$

x	**2**	**3**	**4**
$y(x)$	−1	0	5
$y'(x)$	2	−3	1
$z(x)$	4	2	−3
$z'(x)$	5	−1	7

3 Evaluate $\lim\limits_{x\to\pi} \dfrac{x\cos x + \pi}{x - \pi}$. Show your work.

4 Find a function $h(x)$ that has the given derivative $h'(x)$.

a $h'(x) = 4x^3 \cot x - x^4 \csc^2 x$

b $h'(x) = \dfrac{3\sqrt{x}\csc x + 2x\sqrt{x}\csc x \cot x}{\csc^2 x}$

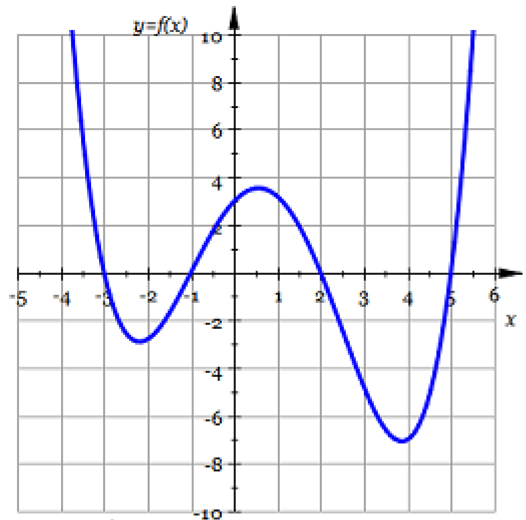
5 The function $f(x) = x^3 + ax^2 + bx + 5$ has a horizontal tangent at the point $(3, -13)$. Show use of calculus to find the constants a and b.

6 Let $h(t)$ be the number of hours per week that a student taking Calculus BC spends on homework and studying, where t is time measured in weeks after September 1. Translate each of the following into a *sentence* about students and homework hours.

a $h(19) = 8$ and $h'(19) = 2$

b $h(36) = 1$ and $h'(36) = -1$

7 Given a graph of $f'(x)$ at right, and $f(3)=-1$.

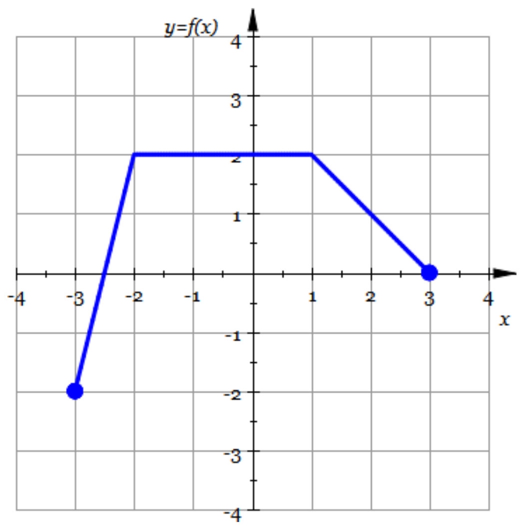


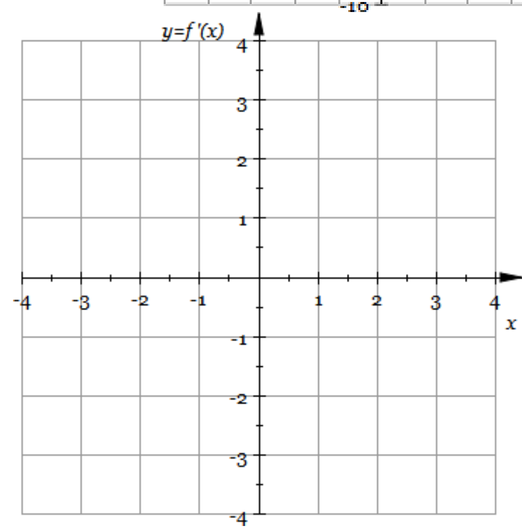
a Write an equation for the line tangent to the graph of $y=f(x)$ at $x=3$.

b Could the line $4x+y=8$ be the line tangent to the graph of $y=f(x)$ at $x=-2$? Why or why not?

c Find the value of $h'(3)$, if $h(x)=\dfrac{f(x)}{x^2+1}$.

8 Given the graph $y=f(x)$ shown, graph the function $y=f'(x)$ on the axes provided.





9 If $f(x)=\dfrac{x}{\tan x}$, then $f'(\pi/4)=$

a 2 **b** $\frac{1}{2}$ **c** $1+\frac{\pi}{2}$ **d** $\frac{\pi}{2}-1$ **e** $1-\frac{\pi}{2}$

10 An equation of the line tangent to the graph of $y=x-\cos x$ at the point $(0,1)$ is

a $y=2x+1$ **b** $y=x+1$ **c** $y=x$ **d** $y=x-1$ **e** $y=0$

11 $\lim\limits_{h\to 0}\dfrac{(-2+h)^4-16}{h}=$

a 64 **b** -32 **c** 16 **d** 0 **e** undefined

12 If u, v, and w are nonzero differentiable functions, then the derivative of $\dfrac{uv}{w}$ is

a $\dfrac{uv'+u'v}{w'}$ **b** $\dfrac{u'v'w-uvw'}{w^2}$ **c** $\dfrac{uvw'-uv'w-u'vw}{w^2}$

d $\dfrac{u'vw+uv'w+uvw'}{w^2}$ **e** $\dfrac{uv'w+u'vw-uvw'}{w^2}$

13 What is the instantaneous rate of change at $x=2$ of $f(x)=\dfrac{x^2-2}{x-1}$?

a -2 **b** $\frac{1}{6}$ **c** $\frac{1}{2}$ **d** 2 **e** 6

Answers to Practice Problems

1a $-4\sin x+15x^{-4}$

1b $\dfrac{-\sin x\left(1+x^2\right)-2x\cos x}{\left(1+x^2\right)^2}$

1c $2\sqrt{3}x^{\sqrt{3}-1}-\cot x\csc x+\frac{2}{3}x^3$

1d $-\csc x+\csc x\sec^2 x$

1e $\frac{45}{4}t^{-9/4}+\tan t\sec t$

2a 3 **2b** $\frac{38}{25}$ **3** -1

4a $x^4\cot x+C$ **4b** $\dfrac{2x\sqrt{x}}{\csc x}+C$

5 $(a,b)=(-4,-3)$

6a "19 weeks after September 1, a student studies 8 hours per week and is increasing his/her study time by 2 hours per week per week."

6b "36 weeks after September 1, a student studies 1 hour per week and is decreasing his/her study time by 1 hour per week per week."

7a $y+1=-5(x-3)$

7b No.

7c $-\frac{11}{25}$

9 E **10** B **11** B **12** E **13** D

Solutions to Practice Problems

1a The derivative of $\cos x$ is $-\sin x$ and $-\dfrac{5}{x^3}$ may be written as $-5x^{-3}$. Then applying the Constant Multiple Rule to the first part and the Power Rule to the second (while realizing that the derivative of the constant π is zero), $f'(x)=-4\sin x+15x^{-4}$.

1b Let $u=\cos x$ and $v=1+x^2$; then $u'=-\sin x$ and $v'=2x$, so applying the Quotient Rule to $p=\dfrac{u}{v}$,

$\dfrac{dp}{dx}=\dfrac{-\sin x\left(1+x^2\right)-2x\cos x}{\left(1+x^2\right)^2}$. You may simplify this if you are so inclined, but it's unnecessary.

1c The derivative of the first term may be found using the Power Rule: $\frac{d}{dx}\left(2x^{\sqrt{3}}\right)=2\sqrt{3}x^{\sqrt{3}-1}$. The derivative of the second term is $\frac{d}{dx}(\csc x)=-\cot x\csc x$, and the derivative of the last term is found by the Power Rule again: $\frac{d}{dx}\left(\frac{x^4}{6}\right)=\frac{d}{dx}\left(\frac{1}{6}x^4\right)=\frac{4}{6}x^3=\frac{2}{3}x^3$. Adding the terms, $h'(x)=2\sqrt{3}x^{\sqrt{3}-1}-\cot x\csc x+\frac{2}{3}x^3$.

1d Let $u=\tan x$ and $v=\csc x$; then $u'=\sec^2 x$ and $v'=-\cot x\csc x$, so applying the Product Rule to $y=uv$, $\frac{dy}{dx}=\tan x(-\cot x\csc x)+\csc x\sec^2 x$. Since $\tan x\cot x=1$, this is equivalent to $\frac{dy}{dx}=-\csc x+\csc x\sec^2 x$.

1e Rewrite the first term as $-9t^{-5/4}$ and apply the Power Rule to get $\frac{d}{dt}\left(-9t^{-5/4}\right)=\frac{45}{4}t^{-9/4}$. The derivative of $\sec t$ is $\tan t\sec t$ and the derivative of $-\frac{\sqrt{5}}{\pi}$ is zero since it's constant, so add everything together to get $\frac{dx}{dt}=\frac{45}{4}t^{-9/4}+\tan t\sec t$.

2a Expand $\frac{d}{dx}(yz)$ with the Product Rule to get $yz'+y'z$ and plug in the relevant values to get 3.

2b Expand $\frac{d}{dx}\left(\frac{z}{y}\right)=\dfrac{yz'-y'z}{y^2}$ and plug in the relevant values to get $\frac{38}{25}$.

3 Recognize this as $\left.\dfrac{d}{dx}(x\cos x)\right|_{x=\pi}$ and note that $\dfrac{d}{dx}(x\cos x)=-x\sin x+\cos x$. Plugging in $x=\pi$ gives -1.

4a Note that this looks like a Product Rule expansion, since $\frac{d}{dx}\left(x^4\right)=4x^3$ and $\frac{d}{dx}(\cot x)=-\csc^2 x$. Thus an antiderivative is $h(x)=x^4\cot x+C$ for some constant C.

4b Note that this looks like a Quotient Rule expansion, and since the denominator has $\csc^2 x$, the denominator of the original function is $\csc x$. In the numerator we have $2x\sqrt{x}$ since its derivative is $3\sqrt{x}$, so an antiderivative is $h(x)=\dfrac{2x\sqrt{x}}{\csc x}+C$ for some constant C.

5 A horizontal tangent means that the tangent's slope is zero. The derivative of the given function with respect to x is $f'(x)=3x^2+2ax+b$, and we know that $f(3)=-13$, so we can set up the system $\begin{cases}3^3+3^2a+3b+5=-13\\ 3\cdot 3^2+2(3)a+b=0\end{cases}$, which has the solution $(a,b)=(-4,-3)$.

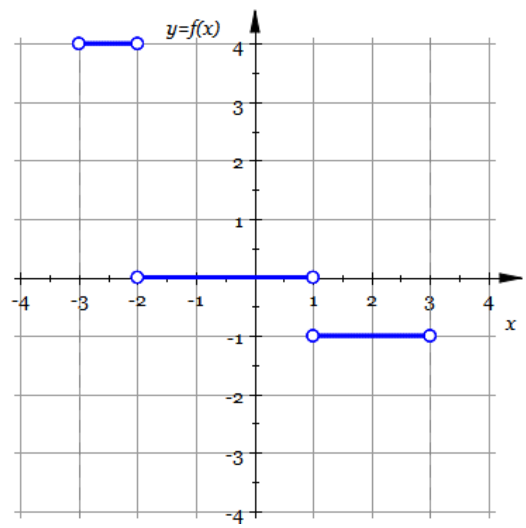
6a and **6b** are hopefully self-explanatory given the answers above.

7a We can read off the graph that $f'(3)=-5$ and are given that $f(3)=-1$, so we can write in point-slope form $y+1=-5(x-3)$.

7b The given line has slope 4, but reading the graph gives $f'(-2)\approx -2.8\neq 4$, so no.

7c The Quotient Rule gives $h'(x)=\dfrac{(x^2+1)f'(x)-2xf(x)}{(x^2+1)^2}$, and the graph shows $f'(3)=-5$. We are given that $f(3)=-1$, so we plug in the necessary values in the Quotient Rule's expression: $h'(3)=\dfrac{(3^2+1)(-5)-2\cdot 3(-1)}{(3^2+1)^2}=-\dfrac{11}{25}$.

8 The function has slope 4 on $(-3,-2)$, slope 0 on $(-2,1)$, and slope -1 on $(1,3)$, and is undefined elsewhere, so the graph of its derivative is as shown to the right.



9 Use the Quotient Rule with $u=x$ and $v=\tan x$. Therefore $u'=1$ and $v'=\sec^2 x$, giving $f'(x)=\dfrac{\tan x-x\sec^2 x}{\tan^2 x}$. Now we can plug in $x=\frac{\pi}{4}$, which gives $f'\left(\frac{\pi}{4}\right)=1-\frac{\pi}{2}$, choice E.

10 $y'=1+\sin x$; plugging in $x=0$, $y'(0)=1+\sin(0)=1$. Since $y(0)=1$, the equation in point-slope form is $y-1=x$, which is equivalent to choice B.

11 Recognize this as $\dfrac{d}{dx}\left(x^4\right)\Big|_{x=-2}$. Apply the Power Rule to find $\dfrac{d}{dx}\left(x^4\right)=4x^3$ and plug in $x=-2$ to get -32, choice B.

12 Let $f=uv$, so $f'=u'v+uv'$. The derivative of the quotient $\frac{f}{w}$ is thus $\dfrac{f'w-fw'}{w^2}$. We substitute what we know for f and f', giving $\dfrac{(u'v+uv')w-uvw'}{w^2}=\dfrac{u'vw+uv'w-uvw'}{w^2}$, which is equivalent to choice E.

13 Use the Quotient Rule, letting $u=x^2-2$ and $v=x-1$; thus $u'=2x$ and $v'=1$, so $f'(x)=\dfrac{2x(x-1)-(x^2-2)}{(x-1)^2}$. Plug in $x=2$ to get 2, choice D.