AP Calculus BC
Review — Derivatives, Part II

Things to Know and Be Able to Do

- Everything that was on the test on the first part of this chapter
- The Chain Rule, used to differentiate composite functions: $\frac{d}{dx}(f \circ g)(x) = g'(x)(f' \circ g)(x)$ or $\frac{d}{dx}(f \circ g)(x) = \frac{df}{dg}\frac{dg}{dx}$
- Use the Chain Rule to differentiate compositions of any number of polynomial, trigonometric, and rational expressions
- Implicit differentiation, an application of the Chain Rule used to differentiate functions defined implicitly; for example, to find the derivative of $x^3 + y^2 = xy$ for y a function of x
- What higher derivatives are and how to find them
- Applications of higher derivatives, including that $v = \frac{dx}{dt}$ and $a = \frac{dv}{dt}$ so $a = \frac{d^2x}{dt^2}$
- How to find expressions for the n^{th} derivative of functions for which it is concisely possible
- How to find derivatives of an arbitrary order for trigonometric functions and their transformations—for example, be able to find $\frac{d^{87}}{dx^{87}}\cos(3x)$ in far fewer than 87 steps
- Solve related rate problems including length dimensions, cross-sectional area, and volume of solids
- Approximate functions with linearization and differentials
- Formulas for quantities related to differentials: $dy = f'(x)\,dx$ and $\Delta y = f(x+\Delta x) - f(x)$
- Error formulas: $\varepsilon = |f(x) - L(x)|$ and $\varepsilon_R = \dfrac{\varepsilon}{f(x)}$ for a linearization $L(x)$ to $f(x)$

Practice Problems

Do not use a calculator on questions 1 and 2. Remember that you need not simplify your answers.

1 Find the indicated derivative of each function. You need not simplify your answers.

a Find $\frac{dy}{dx}$ for $y = (5x^3 - 1)^4 (3 - 4x^2)^7$.

b Find $\frac{dy}{dt}$ for $x^2 y = 36$, $\frac{dx}{dt} = -2$, and $x = 3$.

c Find $\frac{dz}{dx}$ for $z = \tan^4(\cos(3x^5))$.

d Find $\dfrac{d^{43}y}{dx^{43}}$ for $y = 3\sin(5x)$.

2 If $y = (x^3 + 1)^2$, then $\frac{dy}{dx} =$

a $(3x^2)^2$ **b** $2(x^3+1)$ **c** $2(3x^2+1)$ **d** $3x^2(x^3+1)$ **e** $6x^2(x^3+1)$

3 Consider the function $f(x) = (x+1)^k$ where k is a constant.

3a Find the linearization of $f(x)$ at $x = 0$.

3b Use your answer from part **a** to find the linearization of $f(x) = \sqrt[3]{x+1}$ at $x = 0$.

4 Find a rule for $f^{(n)}(x)$ if $f(x) = \dfrac{3}{1-2x}$, showing the steps that lead to your answer.

5 A manufacturer contracts to mint coins for the federal government. What *percentage* error in the radius r of a coin can be tolerated if the coin is to weigh within 0.01% of its ideal weight? Assume that the thickness of the coin does not vary, that the coin is a right circular cylinder, and that the weight of a coin is in direct proportion to its volume. *The volume of a right circular cylinder of radius r and height h is given by* $V = \pi r^2 h$.

6 Consider the curve defined by $2y^3 + 6x^2y - 12x^2 + 6y = 1$.

6a Show that $\dfrac{dy}{dx} = \dfrac{4x - 2xy}{x^2 + y^2 + 1}$.

6b Write an equation of each horizontal tangent line to the curve.

6c The line through the origin with slope -1 is tangent to the curve at P. Find the coordinates of point P.

7 [1984AB5] The volume of a right circular cone is increasing at the rate of 28π cubic units per second. At the instant when the radius r of the cone is 3 units, its volume is 12π cubic units and the radius is increasing at the rate of $\frac{1}{2}$ unit per second. *The volume of a right circular cone of radius r and height h is given by* $V = \frac{1}{3}\pi r^2 h$.

7a When the radius of the cone is 3 units, what is the rate of change of the area of the base?

7b When the radius of the cone is 3 units, what is the rate of change of its height h?

7c When the radius of the cone is 3 units, what is the rate of change of the area of its base with respect to its height h?

8 What is the slope of the line tangent to the curve $3y^2 - 2x^2 = 6 - 2xy$ at the point $(x, y) = (3,2)$?

a 0 **b** $\frac{4}{9}$ **c** $\frac{7}{9}$ **d** $\frac{6}{7}$ **e** $\frac{5}{3}$

9 The table below gives values of $f(x)$, $f'(x)$, $g(x)$, and $g'(x)$ at selected x. If $h(x) = (f \circ g)(x)$, then $h'(1) =$

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
-1	6	5	3	-2
1	3	-3	-1	2
3	1	-2	2	3

a 5 **b** 6 **c** 9 **d** 10 **e** 12

10 If the line tangent to the graph of the function $f(x)$ at the point $(x, f(x)) = (1,7)$ passes through the point $(x, f(x)) = (-2,-2)$, then $f'(1) =$

a -5 **b** 1 **c** 3 **d** 7 **e** undefined

11 If $\frac{dy}{dx} = \sqrt{1 - y^2}$, then $\frac{d^2y}{dx^2} =$

a $-2y$ **b** $-y$ **c** $-\dfrac{y}{\sqrt{1 - y^2}}$ **d** y **e** $\frac{1}{2}$

Answers to Practice Problems

1a $\frac{dy}{dx}=4\left(5x^3-1\right)^3 15x^2\left(3-4x^2\right)^7+\left(5x^3-1\right)^4 7\left(3-4x^2\right)^6(-8x)$ **1b** $\frac{dy}{dt}=\frac{16}{3}$

1c $\frac{dz}{dx}=4\tan^3\left(\cos\left(3x^5\right)\right)\sec^2\left(\cos\left(3x^5\right)\right)\left(-\sin\left(3x^5\right)\right)15x^4$ **1d** $\frac{d^{43}y}{dx^{43}}=-3\cdot 5^{43}\cos(5x)$

2 E **3a** $L(x)=kx+1$ **3b** $L(x)=\frac{1}{3}x+1$

4 $f^{(n)}(x)=\dfrac{3\cdot 2^n n!}{(1-2x)^{n+1}}$

5 0.005%

6a The solution is below.

6b $y\approx 0.1652$ **6c** $(x,y)=\left(-\frac{1}{2},\frac{1}{2}\right)$

7a $\frac{dA}{dt}=3\pi$ units^2/s **7b** $\frac{dh}{dt}=8$ units/s

7c $\frac{dA}{dh}=\frac{3}{8}\pi$ units

8 B **9** D **10** C **11** B

Solutions to Practice Problems

1a Let $u=\left(5x^3-1\right)^4$ and $v=\left(3-4x^2\right)^7$, so $y=uv$. Then the Chain Rule gives $u'=4\left(5x^3-1\right)^3 15x^2$ and $v'=7\left(3-4x^2\right)^6(-8x)$. The Product Rule is then applied to get $y'=u'v+uv'$, so $y'=4\left(5x^3-1\right)^3 15x^2\left(3-4x^2\right)^7+\left(5x^3-1\right)^4 7\left(3-4x^2\right)^6(-8x)$.

1b Implicit differentiation with respect to *t* gives $2x\frac{dx}{dt}y+x^2\frac{dy}{dt}=0$, and the original equation gives $y=4$ when $x=3$, so we plug in the given values: $2(3)(-2)(4)+3^2\frac{dy}{dt}=0$, so $\frac{dy}{dt}=\frac{16}{3}$.

1c Let $f=3x^5$, $g=\cos f$, $h=\tan g$, and $j=h^4$. Therefore $\frac{df}{dx}=15x^4$, $\frac{dg}{df}=-\sin f=-\sin\left(3x^5\right)$, $\frac{dh}{dg}=\sec^2 g=\sec^2\left(\cos\left(3x^5\right)\right)$, and $\frac{dj}{dh}=4h^3=4\tan^3\left(\cos\left(3x^5\right)\right)$. Since $z=(f\circ g\circ h\circ j)(x)$, the Chain Rule gives $\frac{dj}{dx}=\frac{dj}{dh}\frac{dh}{dg}\frac{dg}{df}\frac{df}{dx}=4\tan^3\left(\cos\left(3x^5\right)\right)\sec^2\left(\cos\left(3x^5\right)\right)\left(-\sin\left(3x^5\right)\right)15x^4$.

1d The 43rd derivative of $\sin x$ corresponds to its 3rd derivative. Now $\frac{d^3}{dx^3}(\sin x)=-\cos x$, but we also need to apply the Chain Rule 43 times and multiply by 3, so $\frac{d^{43}}{dx^{43}}(3\sin(5x))=-3\cdot 5^{43}\cos(5x)$.

2 Apply the chain rule to get $y'=2\left(x^3+1\right)3x^2=6x^2\left(x^3+1\right)$, which is choice E.

3a $f'(x)=k(x+1)^{k-1}$, so $f'(0)=k(1)^{k-1}=k$. Then $f(0)=1$, so the linearization is $L(x)=kx+1$.

3b This is simply the above case with $k=\frac{1}{3}$, so we plug that in to get $L(x)=\frac{1}{3}x+1$.

4 Consider the table at right. Hopefully, the pattern is apparent: $f^{(n)}(x)=\dfrac{3\cdot 2^n n!}{(1-2x)^{n+1}}$.

n	$f^{(n)}(x)$	
1	$\frac{-3}{(1-2x)^2}(-2)$	$=\frac{3\cdot 2}{(1-2x)^2}$
2	$\frac{-2\cdot 3\cdot 2}{(1-2x)^3}(-2)$	$=\frac{3\cdot 2^2\cdot 2}{(1-2x)^3}$
3	$\frac{-3\cdot 3\cdot 2^2\cdot 2}{(1-2x)^4}(-2)$	$=\frac{3\cdot 2^3\cdot 6}{(1-2x)^4}$
4	$\frac{-4\cdot 3\cdot 2^3\cdot 6}{(1-2x)^5}(-2)$	$=\frac{3\cdot 2^4\cdot 24}{(1-2x)^5}$

5 Let *w* signify the weight of the object. Since *w* is proportional to volume, let *k* be the constant of proportionality; then $w=k\pi r^2h$. Then $\frac{dw}{dr}=2k\pi rh\Rightarrow dw=2k\pi rh\,dr$, and we are given a relative error of 0.0001. This means that $\frac{dw}{w}\le 0.0001$. Plugging in our known values for dw and w, $\frac{2\cancel{k\pi r}\cancel{h}\,dr}{\cancel{k\pi r^2}\cancel{h}}=2\frac{dr}{r}$. So $2\frac{dr}{r}\le 0.0001\Rightarrow\frac{dr}{r}\le 0.00005$, or 0.005%.

6a The terms on the left side of the equation are differentiated implicitly as follows: $\frac{d}{dx}\left(2y^3\right)=6y^2\frac{dy}{dx}$, $\frac{d}{dx}\left(6x^2y\right)=12xy+6x^2\frac{dy}{dx}$ using the Product Rule, $\frac{d}{dx}\left(-12x^2\right)=-24x$, and $\frac{d}{dx}(6y)=6\frac{dy}{dx}$. On the right side of the equation, $\frac{d}{dx}(1)=0$. Therefore $6y^2\frac{dy}{dx}+12xy+6x^2\frac{dy}{dx}-24x+6\frac{dy}{dx}=0$, or $\frac{dy}{dx}\left(6x^2+6y^2+6\right)=24x-12xy$. Dividing, $\frac{dy}{dx}=\frac{24x-12xy}{6x^2+6y^2+6}=\frac{\cancel{6}\left(4x-2xy\right)}{\cancel{6}\left(x^2+y^2+1\right)}=\frac{4x-2xy}{x^2+y^2+1}$.

6b We want $\frac{dy}{dx}=0$, meaning that $4x-2xy=0$. Factoring gives $2x(2-y)=0$, so the Zero Product Property tells us that $x=0$ or $y=2$. If $x=0$, then $2y^3+6y=1\Rightarrow y\approx 0.1652$. If $y=2$, then $16+12x^2-12x^2+12=1\Rightarrow 28=1$ which obviously has no solution, so the only horizontal tangent is $y\approx 0.1652$.

6c We have two equations, which can be set up as a system: $\begin{cases} y=-x \\ \frac{4x-2xy}{x^2+y^2+1}=-1 \end{cases}$. Substituting the first into the second gives $\frac{4x-2x(-x)}{x^2+(-x)^2+1}=\frac{4x+2x^2}{2x^2+1}=-1$. Multiplying by the denominator gives $4x+2x^2=-2x^2-1$, which can be solved in any of a number of ways to yield $x=-\frac{1}{2}$, and therefore $y=\frac{1}{2}$. Thus $(x,y)=\left(-\frac{1}{2},\frac{1}{2}\right)$.

7a We know that $\frac{dV}{dt}=28\pi$ when $r=3$, $V=12\pi$, $h=4$, and $\frac{dr}{dt}=\frac{1}{2}$. Now since $A=\pi r^2$, we can differentiate with respect to time to find $\frac{dA}{dt}=2\pi r\frac{dr}{dt}$. Now we plug in the necessary values: $\frac{dA}{dt}=2\pi(3)\left(\frac{1}{2}\right)=3\pi$ units^2/s.

7b Since $V=\frac{1}{3}\pi r^2 h$, we differentiate using the Product Rule (since both r and h depend on t) to find $\frac{dV}{dt}=\frac{2}{3}\pi rh\frac{dr}{dt}+\frac{1}{3}\pi r^2\frac{dh}{dt}$. Then we plug in the known values from the previous part, getting $28\pi=\frac{2}{3}\pi(3)(4)\left(\frac{1}{2}\right)+\frac{1}{3}\pi(3)^2\frac{dh}{dt}$. Therefore $\frac{dh}{dt}=8$ units/s.

7c $\frac{dA}{dh}=\frac{\frac{dA}{dt}}{\frac{dh}{dt}}=\frac{3\pi}{8}$ units.

8 Implicit differentiation gives $6y\frac{dy}{dx}-4x=-2x\frac{dy}{dx}-2y$, so $\frac{dy}{dx}=\frac{4x-2y}{2x+6y}=\frac{2x-y}{x+3y}$. Plugging in $(x,y)=(3,2)$ gives $\frac{dy}{dx}=\frac{4}{9}$, choice B.

9 The Chain Rule says $h'(1)=(f'\circ g)(1)g'(1)$. The table shows that $g(1)=-1$, so we need to find $f'(-1)=5$. Therefore $h'(1)=5\cdot 2=10$, choice D.

10 We know two points on the line: $(x_0,y_0)=(1,7)$ and $(x_1,y_1)=(-2,-2)$. This is enough to determine the slope of the line: $m=\frac{y_0-y_1}{x_0-x_1}=\frac{7-(-2)}{1-(-2)}=3$, choice C.

11 The Chain Rule gives $\frac{d^2y}{dx^2}=\frac{-2y\frac{dy}{dx}}{2\sqrt{1-y^2}}=-\frac{\cancel{2}y\cancel{\sqrt{1-y^2}}}{\cancel{2}\cancel{\sqrt{1-y^2}}}=-y$, choice B.