AP Calculus BC
Review — Applications of Derivatives (Chapter 4)

Things to Know and Be Able to Do

- Definitions of the following in terms of derivatives, and how to find them: critical point, global minima/maxima, local (relative) minima/maxima, inflection point
- The Mean Value Theorem, including Rolle's Theorem, and how to apply each of them
- The First Derivative Test and the Second Derivative Test
- Interpretations of concavity in terms of derivatives
- Given any one of f, f', and f'', find characteristics of the others and be able to sketch a graph of $y = f(x)$
- Horizontal and slant (oblique) asymptotes and how to find them, including polynomial long division
- Optimization problems including the use of trigonometry (be able to justify your answers, usually by using the Second Derivative Test to show that the relevant point is a minimum when you want a minimum or a maximum when you want a maximum)
- Newton's Method including iterations thereof *and* its derivation
- Basic antiderivatives using the Sum, Difference, Constant Multiple, Product, Quotient, Power, and Chain Rules (don't forget +C!)

Practice Problems

Problems 1–3 are to be done without the use of a calculator. Problems 4–12 may be done with a calculator.

1 Find the most general antiderivative for each function given.

a $f(x) = \sec x \tan x - \cos(2x) + \csc^2 x - 2$

b $f(x) = 4x^5 - 5\sqrt[3]{x} + \dfrac{1}{x^2} - \dfrac{3}{\sqrt[5]{x^7}}$

c $f(x) = 4\tan^3 x \sec^2 x$

d $f(x) = x^3 \cos x + 3x^2 \sin x$

2 (1990AB1) A particle, initially at rest, moves along the x-axis so that its acceleration at any time $t \geq 0$ is given by $a(t) = 12t^2 - 4$. The position of the particle when $t = 1$ is $x(1) = 3$.

a Find the value(s) of t for which the particle is at rest.

b Write an expression for the position $x(t)$ of the particle at any time $t \geq 0$.

c Find the total distance traveled by the particle from $t = 0$ to $t = 2$.

3 The function $f'(x)$ with domain $[-7,7]$ is shown at right. For all parts, justify your answer.

a On what interval(s) is $f(x)$ increasing?

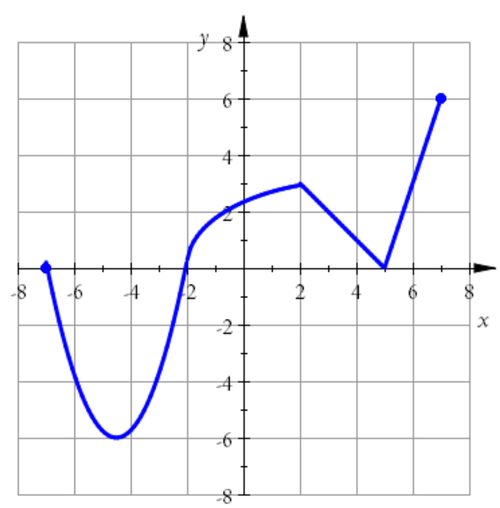
b On what interval(s) is $f(x)$ concave down?

c On what interval(s) is $f''(x)$ positive?

d At what value(s) of x does $f(x)$ have a relative minimum?

e At what value(s) of x does $f(x)$ have a point of inflection?

f If $l(x)$ is the tangent line to the graph of $f(x)$ at the point where $x = -3$, which is larger—$l(-2)$ or $f(-2)$? Explain.



4 (1990BC3) Let $f(x)=12-x^2$ for $x\geq 0$ and $f(x)\geq 0$.

a The line tangent to the graph of *f* at the point $(k, f(k))$ intercepts the *x*-axis at $x=4$. What is the value of *k*? Show all of your work.

b An isosceles triangle whose base is the interval from $(0,0)$ to $(c,0)$ has its vertex on the graph of *f*. For what value of *c* does the triangle have maximum area? Justify your answer.

5 Find a value of *b* for which the function $y=x^4+bx^2+8x+1$ has both a horizontal tangent line and a point of inflection at the same value of *x*. Show all of your work.

6 A crate, open at the top, has four vertical sides, a square bottom with side length *x*, a height *h*, and a volume of 4 m^3. If the crate has the least possible surface area, show use of calculus to find its dimensions. Justify your answer.

7 The function $f(x)=x^{2/3}$ on $x\in[-8,8]$ does not satisfy the conditions of the Mean Value Theorem because

a $f(0)$ is not defined.

b $f(x)$ is not continuous on $[-8,8]$.

c $f'(-1)$ does not exist.

d $f(x)$ is not defined for $x<0$.

e $f'(0)$ does not exist.

8 Let *f* be a function that is differentiable on the open interval $x\in(1,10)$. If $f(2)=-5$, $f(5)=5$, and $f(9)=-5$, which of the following must be true?

I *f* has at least two zeros.

II The graph of *f* has at least one horizontal tangent.

III For some *c*, $2<c<5$, $f(c)=3$.

a None **b** I only **c** I and II only **d** I and III only **e** I, II, and III

9 If Newton's Method is used to approximate the positive root of $f(x)=x^2+2x-5$, starting with $x_0=2$, then the value of x_2 is

a $\frac{1}{4}$ **b** $\frac{2}{5}$ **c** $\frac{29}{20}$ **d** $\frac{3}{2}$ **e** $\frac{33}{20}$

10 Let $f(x)=x^3-3x+1$. In using Newton's Method to approximate the root of *f* in $[1,2]$, which equation below will find this root (with initial estimate $x_0=1.5$)?

a $x_{n+1}=x_n-\dfrac{3x_n^{\,2}-3}{x_n^{\,3}-3x_n+1}$

b $x_{n+1}=x_n+\dfrac{3x_n^{\,2}-3}{x_n^{\,3}-3x_n+1}$

c $x_{n+1}=x_n-\dfrac{x_n^{\,3}-3x_n+1}{3x_n^{\,2}-3}$

d $x_{n+1}=x_n+\dfrac{x_n^{\,3}-3x_n+1}{3x_n^{\,2}-3}$

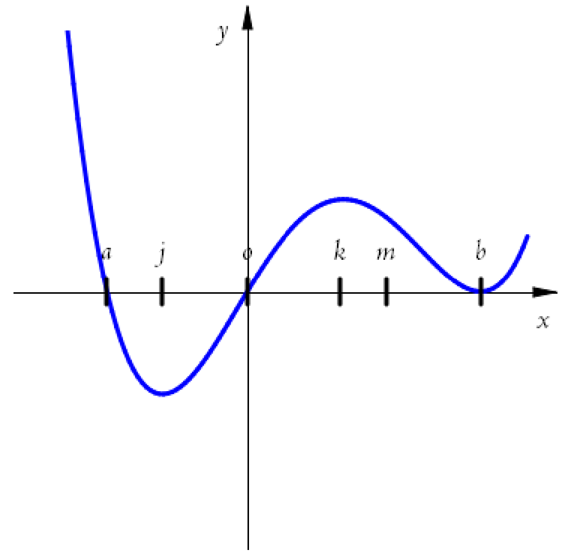
e $x_{n+1}=x_n+\dfrac{x_n^{\,3}-3x_n+1}{3x_n^{\,2}+3}$

11 Which of the following represents the oblique asymptote for $f(x)=\dfrac{3x^2-5x+2}{x-4}$?

a $y=3x$ **b** $y=3x+4$ **c** $y=3x-17$ **d** $y=3x-5$ **e** $y=3x+7$

12 The second derivative of the function f is given by $f''(x)=x(x-a)(x-b)^2$. The graph of f'' is shown at right. For what values of x does the graph of f have a point of inflection?

a 0 and a
b 0 and m only
c b and j only
d 0, a, and b
e j and k



Answers

1a $F(x)=\sec x-\frac{1}{2}\sin(2x)-\cot x-2x+C$

1b $F(x)=\frac{2}{3}x^6-\frac{15}{4}x^{4/3}-\frac{1}{x}+\frac{15}{2}x^{2/5}+C$

1c $F(x)=\tan^4 x+C$

1d $F(x)=x^3\sin x+C$

2a $t=0$ and $t=1$

2b $x(t)=t^4-2t^2+4$

2c 8

3a $(-2,5)\cup(5,7)$, **3b** $(-7,-5)\cup(2,5)$

3c $(-5,2)\cup(5,7)$, **3d** $x=-2$

3e $x\in\{-5,2,5\}$, **3f** $f(-2)$

4a $k=2$, **4b** $c=4$

5 $b=-6$

6 $x=2$ and $h=1$

7 e, **8** e, **9** c, **10** c, **11** e, **12** a

Solutions

1a The first term is the derivative of $\sec x$; the second (by the Chain Rule), the derivative of $-\frac{1}{2}\sin(2x)$; the third, the derivative of $-\cot x$; and the -2 simply needs to be multiplied by x. Don't forget the constant. $F(x)=\sec x-\frac{1}{2}\sin(2x)-\cot x-2x+C$.

1b Rewrite the second through fourth terms as $-5x^{1/3}+x^{-2}-3x^{-7/5}$ and apply the Power Rule for Antiderivatives to the whole thing to get $F(x)=\frac{2}{3}x^6-\frac{15}{4}x^{4/3}-\frac{1}{x}+\frac{15}{2}x^{2/5}+C$.

1c This is the output of the Chain Rule applied to $\tan^4 x$. Including the constant, $F(x)=\tan^4 x+C$.

1d This is the output of the Product Rule applied to x^3 and $\sin x$. Therefore $F(x)=x^3\sin x+C$.

2a Given $a(t)$, we can find $v(t)$ by antidifferentiating $a(t)$. We get $v(t)=4t^3-4t+C$, and since we know that $v(0)=0$, we can solve $0=4(0)^3-4(0)+C$ for $C=0$. This means $v(t)=4t^3-4t$. Solving $4t^3-4t=0$ gives $t\in\{0,1\}$.

2b Since we have $v(t)$ from part **a**, we can find $x(t)$ by antidifferentiating $v(t)$. This gives $x(t)=t^4-2t^2+C_1$. Since we've been told that $x(1)=3$, we can solve $3=1^4-2(1)^2+C_1$ for $C_1=4$, meaning that $x(t)=t^4-2t^2+4$.

3a The function is increasing when $f'(x)>0$, which is on $(-2,5)\cup(5,7)$.

3b The function is concave down when $f'(x)$ is decreasing, which is on $(-7,-5)\cup(2,5)$.

3c The second derivative is positive when $f'(x)$ is increasing, which is on $(-5,2)\cup(5,7)$.

3d Relative minima occur at values of x such that $f'(x)=0$ and $f'(x)$ is changing from negative to positive. This situation occurs only at $x=-2$.

3e Points of inflection occur at values of x such that $f'(x)$ has a relative extreme. This situation occurs at $x=-5$, $x=2$, and $x=5$.

3f The function is concave up from -3 to -2 since $f'(x)$ is increasing on that interval. Therefore the tangent line at an x value will, at a higher value of x still on that interval, have a smaller y-value than the function itself will. Therefore $f(-2)>l(-2)$.

4a $f'(x)=-2x$ and x-intercepts occur at values of x such that $f(x)=0$. The tangent line to f at $(k,f(k))$ is $y-f(k)=-2k(x-k)$, and plugging in $(x,y)=(4,0)$, we have $-f(k)=2k(4-k)$. Since we know $f(k)=12-k^2$, the equation is $-12+k^2=-2k(4-k)$, which is solved for $k\in\{2,6\}$. However, $f(6)=-24$ and the function is only defined when its output is nonnegative, so $k=2$.

4b The isosceles triangle has a base of length c and the x-coordinate of the third point is halfway between 0 and c; that is, $\frac{c}{2}$. Therefore the triangle has area $A=\frac{1}{2}c\cdot f\left(\frac{c}{2}\right)$. Plugging in $\frac{c}{2}$ to f gives $A=\frac{1}{2}c\left(12-\frac{c^2}{4}\right)=6c-\frac{1}{8}c^3$. We wish to maximize this, so $\frac{dA}{dc}=6-\frac{3}{8}c^2=0$ gives $c=\pm4$, of which only $c=4$ is relevant. We justify this by finding $\frac{d^2A}{dc^2}=-\frac{3}{4}c$ and plugging in $c=4$ to find $\left.\frac{d^2A}{dc^2}\right|_{c=4}=-3$, and since -3 is negative, this is a relative maximum.

5 The horizontal tangent means that at some x, $y'(x)=0$, and the point of inflection means that at (the same) x, $y''(x)=0$. Since $y'=4x^3+2bx+8$ and $y''=12x^2+2b$, we can set up the system $\begin{cases} y=x^4+bx^2+8x+1 \\ 0=4x^3+2bx+8 \\ 0=12x^2+2b \end{cases}$ which solves to $(x,y)=(1,4)$, and more importantly, $b=-6$.

6 The volume is $V=x^2h$ and since $V=4$, we know $h=\frac{4}{x^2}$. The surface area is $A=x^2+4hx$, but using our known h, $A=x^2+4\left(\frac{4}{x^2}\right)x=x^2+\frac{16}{x}$. We wish to minimize this, so $\frac{dA}{dx}=2x-\frac{16}{x^2}=0$ gives $x=2$. We justify this by finding $\frac{d^2A}{dx^2}=2+\frac{32}{x^3}$. Plugging in $x=2$ gives $\left.\frac{d^2A}{dx^2}\right|_{x=2}=6$ and since $6>0$, this is a relative minimum.

7 Option **a** is invalid because $f(0)$ exists (and is equal to 0). Option **b** is not true. The derivative is $f'(x)=\frac{2}{3}x^{-1/3}$, so **c** is invalid; $f'(-1)=-\frac{2}{3}$. Option **d** is also not true. This leaves option **e**, which is true: finding $f'(0)$ would require dividing by zero. Therefore **e** is correct.

8 I is true because, since $f(2)=-5$ and $f(5)=5$, f must be zero somewhere between 2 and 5 by the Intermediate Value Theorem; likewise, since $f(5)=5$ and $f(9)=-5$, f must be zero somewhere between 5 and 9. (Note that f being differentiable means that f is continuous and thus meets the condition of the Intermediate Value Theorem.) II is true by Rolle's Theorem: since $f(2)=f(9)$, there exists some c between 2 and 9 such that $f'(c)=0$.

III is true by the Intermediate Value Theorem: since $f(2)=-5$ and $f(5)=5$, between 2 and 5, *f* assumes every value between −5 and 5. Since I, II, and III are all true, the answer is **e**.

9 Newton's Method gives $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$. Since $f'(x)=2x+2$, $x_1 = 2 - \frac{2^2+2(2)-5}{2(2)+2} = \frac{3}{2}$ and $x_2 = \frac{3}{2} - \frac{f(\frac{3}{2})}{f'(\frac{3}{2})} = \frac{3}{2} - \frac{(\frac{3}{2})^2+2(\frac{3}{2})-5}{2(\frac{3}{2})+2} = \frac{3}{2} - \frac{1}{20} = \frac{29}{20}$. This is choice **c**.

10 Newton's Method gives $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. Since $f'(x)=3x^2-3$, $x_{n+1} = x_n - \frac{x^3-3x_n+1}{3x_n{}^2-3}$, choice **c**.

11 Carry out the long division as follows:

$$\begin{array}{r} 3x+7 \text{ R}30 \\ x-4\overline{)3x^2-5x+2} \\ \underline{-(3x^2-12x)} \\ 7x+2 \\ \underline{-(7x-28)} \\ 30 \end{array}$$

The quotient is $3x+7$, which corresponds to option **e**.

12 Points of inflection occur at values of *x* such that $f''(x)=0$ *and* is changing sign. This situation is at 0 and *a* only.