AP Calculus BC
Review — Integrals (Chapter 5)

Things to Know and Be Able to Do

- Know what a definite integral represents, including the concept of signed area
- Basic applications of definite integrals, including the relationships between position, speed, and acceleration
- Riemann sum approximations for $\int_a^b f(x)\,dx$ using n subintervals:
  - Left sum: $\frac{b-a}{n}\sum_{i=0}^{n-1} f\left(a+\frac{i(b-a)}{n}\right)$
  - Right sum: $\frac{b-a}{n}\sum_{i=1}^{n} f\left(a+\frac{i(b-a)}{n}\right)$
  - Midpoint sum: $\frac{b-a}{n}\sum_{i=0}^{n-1} f\left(a+\frac{(i+\frac{1}{2})(b-a)}{n}\right)$ or $\frac{b-a}{n}\sum_{i=1}^{n} f\left(a+\frac{(i-\frac{1}{2})(b-a)}{n}\right)$
  - Trapezoidal sum: average of the left and right sums
- Know what each part of those expression means and why it works
- The Riemann sum can be turned into a definite integral by taking $\lim_{n\to\infty}$
- Understand that these definite integrals *can* be evaluated from "first principles"; that is, expressions like $\sum_{i=1}^{n} i=\frac{n(n+1)}{2}$ and $\sum_{i=1}^{n} i^2=\frac{n(n+1)(2n+1)}{6}$. However, you don't necessarily need to be able to do this. The first of those expressions is definitely a good thing to know, though.
- Properties of the definite integral:
  - $\int_a^b f(x)\,dx=-\int_b^a f(x)\,dx$
  - $\int_a^a f(x)\,dx=0$
  - $\int_a^b c\,dx=c(b-a)$
  - $\int_a^b \left(f(x)+g(x)\right)dx=\int_a^b f(x)\,dx+\int_a^b g(x)\,dx$
  - $\int_a^b c\,f(x)\,dx=c\int_a^b f(x)\,dx$
  - $\int_a^c f(x)\,dx=\int_a^b f(x)\,dx+\int_b^c f(x)\,dx$ if b is between a and c
  - Most of these are common sense. Use common sense.
- The Fundamental Theorem of Calculus, Part One: $\frac{d}{dx}\int_a^x f(t)\,dt=f(x)$ for any constant a
  - Understand the proof and how to apply this, including if the upper limit isn't just x but is a function
- The Fundamental Theorem of Calculus, Part Two: $\int_a^b f'(x)\,dx=f(b)-f(a)$
  - Understand the proof and how to apply this. Especially because 80% of the rest of the course is applying this.
- Indefinite integrals are just antiderivatives in disguise. They're notated like definite integrals, but without limits. Don't forget $+C$.
- u-substitution: Best understood by example; see pages 360–363.

Practice Problems

Problems 1–5 may be done with the use of a calculator; when this test was originally administered, there was a 20-minute time limit on that section. Problems 6–12 should be done without a calculator.

1 Rocket *A* has positive velocity $v(t)$ after being launched upward from an initially height of 0 ft at time $t = 0$. The velocity of the rocket is recorded for selected values of *t* over the interval $0 \le t \le 80$ s, as shown in the table below.

t [s]	0	10	20	30	40	50	60	70	80
$v(t)$ [ft/s]	5	14	22	29	35	40	44	47	49

a Find the average acceleration of rocket *A* over the time interval $0 \le t \le 80$ s. Indicate units of measure.

b Use a midpoint Riemann sum with 4 subintervals of equal length to approximate $\int_{0\text{ s}}^{80\text{ s}} v(t)\,dt$. Indicate units of measure, and explain the meaning of the integral in terms of the rocket's flight.

c Rocket *B* is launched upward with an acceleration of $a(t) = \frac{3}{\sqrt{t+1}}$ ft/s^2. At time $t = 0$ s, the initial height of the rocket is 0 ft and the initial velocity is 2 ft/s. Which of the two rockets is traveling faster at time $t = 80$ s? Justify your answer.

2 A car travels on a straight track. During the time interval $0 \le t \le 60$ s, the car's velocity *v*, measured in ft/s, and acceleration *a*, measured in ft/s^2, are continuous functions. The table below shows selected values of these functions.

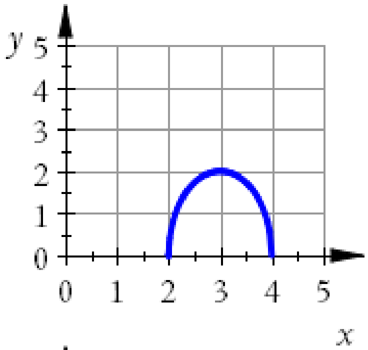
t [s]	0	15	25	30	35	50	60
$v(t)$ [ft/s]	–20	–30	–20	–14	–10	0	10
$a(t)$ [ft/s^2]	1	5	2	1	2	4	2

a Approximate $\int_{30\text{ s}}^{60\text{ s}} v(t)\,dt$ with a left rectangle approximation using the three subintervals determined by the table. Give appropriate units with your answer and explain the meaning of the integral in terms of the car's motion. *Note: the subintervals are not of equal width.*

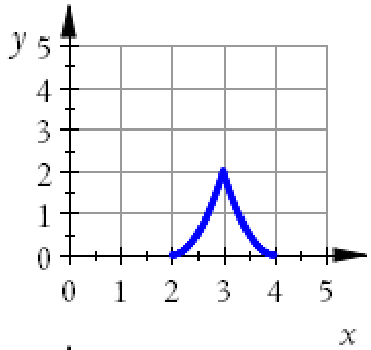
b Find the exact value of $\int_{0\text{ s}}^{30\text{ s}} a(t)\,dt$. Give appropriate units with your answer and explain the meaning of the integral in terms of the car's motion.

3 On the closed interval $x \in [2,4]$, which of the following could be the graph of a function *f* with the property that $\frac{1}{4-2}\int_2^4 f(x)\,dx = 1$?

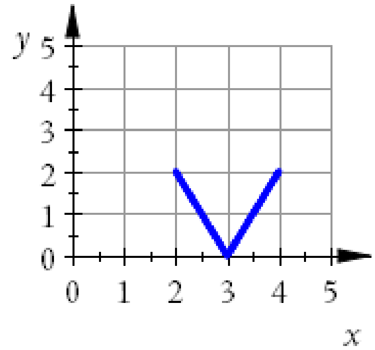
a



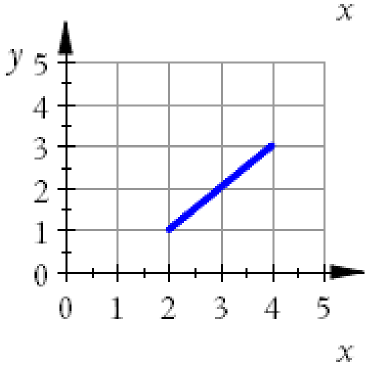
b



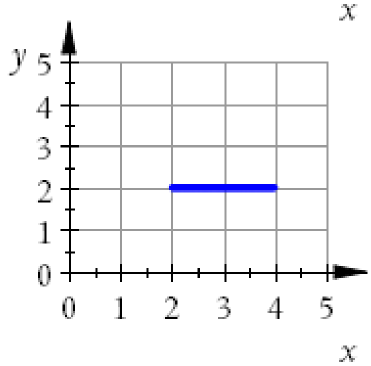
c



d



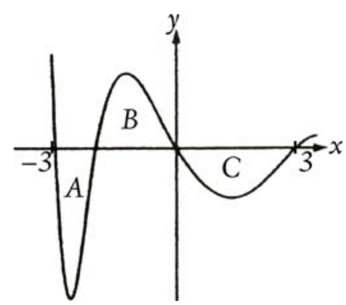
e



4 If *n* is a positive integer, then $\lim_{n\to\infty}\frac{3}{n}\left[\left(\frac{3}{n}\right)^2+\left(\frac{6}{n}\right)^2+\left(\frac{9}{n}\right)^2+\cdots+\left(\frac{3n}{n}\right)^2\right]$ can be expressed as

a $\int_0^1 3x^2\,dx$ **b** $3\int_0^1\left(\frac{1}{x}\right)^2 dx$ **c** $\int_0^3\left(\frac{1}{x}\right)^2 dx$ **d** $\int_0^3 x^2\,dx$ **e** $3\int_0^3 x^2\,dx$

5 The regions *A*, *B*, and *C* in the figure at right are bounded by the graph of the function *f* and the *x*-axis. If the area of each region is 2, what is the value of $\int_{-3}^{3}\left(f(x)+1\right)dx$?



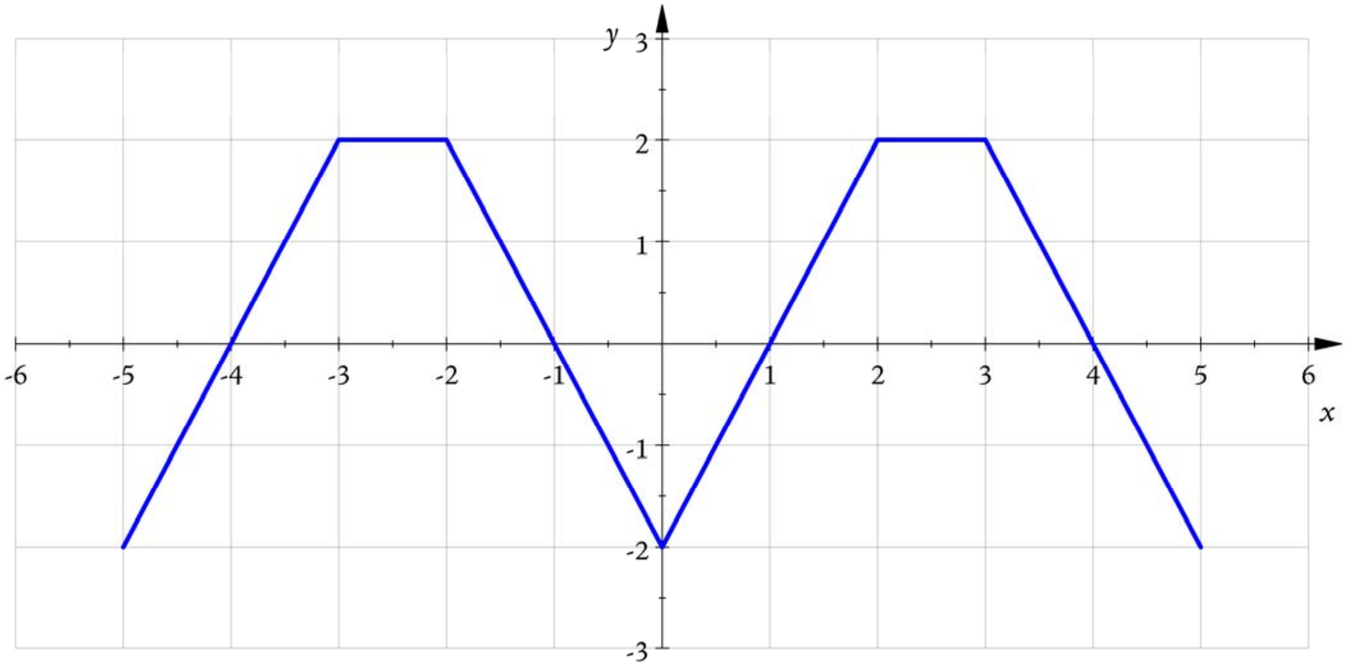
a −2 **b** −1 **c** 4 **d** 7 **e** 12

6 Evaluate each expression.

a $\int 4\cos x\,(3+4\sin x)^{2005}\,dx$ **b** $\int \frac{3^{\sqrt{3-x^2}}\,x}{\sqrt{3-x^2}}\,dx$ **c** $\int \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx$

d $\int_{-3}^{3}\left(4+\sqrt{9-x^2}\right)dx$ **e** $\int_1^e \frac{\sin(\ln x)}{x}\,dx$

7 The graph of the function *f* shown below consists of six line segments. Let *g* be given by $g(x)=\int_0^x f(t)\,dt$.



Graph of *f*

a Find $g(4)$, $g'(4)$, and $g''(4)$.

b Does *g* have a relative minimum, a relative maximum, or neither at $x=1$? Justify your answer.

c Suppose that *f* is defined for all real numbers *x* and is periodic with a period of length 5. The graph above shows two periods of *f*.

  i Given that $g(5)=2$, find $g(10)$.

  ii Write an equation for the line tangent to the graph of *g* at the point where $x=108$.

8 Given the Riemann sum $\sum_{k=1}^{20}\frac{1}{5}\left(2+\frac{k}{5}\right)^2$,

a If this Riemann sum represents a right rectangle approximation, what area does the sum represent? Be specific.

b Assuming that the sum is a right rectangle sum, write an expression in summation notation that represents an approximation using *n* rectangles.

c Write a definite integral that represents the exact area expressed in part **a**. Do not evaluate it.

9 Using the substitution $u=2x+1$, $\int_0^2 \sqrt{2x+1}\,dx$ is equivalent to

a $\frac{1}{2}\int_{-1/2}^{1/2}\sqrt{u}\,du$ **b** $\frac{1}{2}\int_0^2\sqrt{u}\,du$ **c** $\frac{1}{2}\int_1^5\sqrt{u}\,du$ **d** $\int_0^2\sqrt{u}\,du$ **e** $\int_1^5\sqrt{u}\,du$

10 $\frac{d}{dx}\left(\int_0^{x^2}\sin\left(t^3\right)dt\right)=$

a $-\cos\left(x^6\right)$ **b** $\sin\left(x^3\right)$ **c** $\sin\left(x^6\right)$ **d** $2x\sin\left(x^3\right)$ **e** $2x\sin\left(x^6\right)$

11 $\int x^2\cos\left(x^3\right)dx=$

a $-\frac{1}{3}\sin\left(x^3\right)+C$ **b** $\frac{1}{3}\sin\left(x^3\right)+C$ **c** $-\frac{x^3}{3}\sin\left(x^3\right)+C$ **d** $\frac{x^3}{3}\sin\left(x^3\right)+C$ **e** $\frac{x^3}{3}\left(\sin\frac{x^4}{4}\right)+C$

Answers

1a $0.55\ \text{ft/s}^2$ **1b** 2600 ft **1c** *B*
2a -220 ft **2b** $6\ \text{ft/s}$
3 c **4** d **5** c
6a $\frac{1}{2006}(3+4\sin x)^{2006}+C$
6b $-\dfrac{3^{\sqrt{3-x^2}}}{\ln 3}+C$ **6c** $\frac{1}{2}\left(\sin^{-1}x\right)^2+C$
6d $24+\frac{9}{2}\pi$ **6e** $-\cos 1+1$

7a $g(4)=3;\ g'(4)=0;\ g''(4)=-2$
7b relative minimum
7ci 4
7cii $y-42=2(x-108)$

8a an approximation to the area under the graph of $f(x)=x^2$ on the interval $x\in[2,6]$ using 20 right rectangles
8b $\dfrac{4}{n}\sum_{i=1}^{n}\left(2+\dfrac{4i}{n}\right)^2$ **8c** $\int_2^6 x^2dx$
9 c **10** e **11** b

Solutions

1a The average acceleration is given by $\bar{a}=\dfrac{\Delta v}{\Delta t}=\dfrac{(49\ \text{m/s})-(5\ \text{m/s})}{(80\ \text{s})-(0\ \text{s})}=\dfrac{44}{80}\ \text{m/s}^2=0.55\ \text{m/s}^2$.

1b The four subintervals are $t\in(0\ \text{s},20\ \text{s})$, $t\in(20\ \text{s},40\ \text{s})$, $t\in(40\ \text{s},60\ \text{s})$, and $t\in(60\ \text{s},80\ \text{s})$. The midpoints of these are $t=10$, 30, 50, and 70, respectively. Each subinterval has length 20, and therefore the midpoint approximation is given by $20\big(v(10\ \text{s})+v(30\ \text{s})+v(50\ \text{s})+v(70\ \text{s})\big)=20(14\ \text{ft}+29\ \text{ft}+40\ \text{ft}+47\ \text{ft})=2600\ \text{ft}$. This is the distance traveled by the rocket in its first 80 s of flight.

1c We know from the table that $v_A(80\ \text{s})=49\ \text{ft/s}$. To find Rocket *B*'s velocity at 80 s, we take $\int_{0\ \text{s}}^{80\ \text{s}}\dfrac{3}{\sqrt{t+1}}dt$. An antiderivative of the integrand (with respect to *t*) is $6\sqrt{t+1}$. This evaluated at $t=80$ s, is $54\ \text{ft/s}$, and at $t=0$ s, it is $6\ \text{ft/s}$. The difference is $(54\ \text{ft/s})-(6\ \text{ft/s})=48\ \text{ft/s}$, meaning that Rocket *B*'s speed has *increased* by $48\ \text{ft/s}$ in its first 80 s of flight. However, we are also told that $v_B(0\ \text{s})=2\ \text{ft/s}$, so we add this to the $48\ \text{ft/s}$ to get $v_B(80\ \text{s})=50\ \text{ft/s}$. Therefore $v_B(80\ \text{s})>v_A(80\ \text{s})$.

2a The three subintervals on the given interval are $t\in(30\ \text{s},35\ \text{s})$, $t\in(35\ \text{s},50\ \text{s})$, and $t\in(50\ \text{s},60\ \text{s})$. The widths of these are 5 s, 15 s, and 10 s, respectively, and their left points evaluate to $v(30\ \text{s})=-14\ \text{ft/s}$, $v(35\ \text{s})=-10\ \text{ft/s}$, and $v(50\ \text{s})=0$ s, respectively. Therefore the left Riemann sum approximation is given by $(5\ \text{s})(-14\ \text{ft/s})+(15\ \text{s})(-10\ \text{ft/s})+(10\ \text{s})(0\ \text{ft/s})=-220\ \text{ft}$. This is the change in position of the car from $t=30$ s to $t=60$ s.

2b Because $\int a(t)dt=v(t)$, the Fundamental Theorem of Calculus, Part Two allows us to evaluate $\int_{0\ \text{s}}^{30\ \text{s}}a(t)dt$ by evaluating $v(30\ \text{s})-v(0\ \text{s})=(-14\ \text{ft/s})-(-20\ \text{ft/s})=6\ \text{ft/s}$.

3 The expression means that the average value of $f(x)$ on $x\in[2,4]$ is 1. The only function for which that could possibly be true is **c**, since the average value of the function depicted in **a** seems to be in the vicinity of $\frac{5}{3}$; in **b**, $\frac{3}{4}$; and in **d** and **e**, it's pretty clearly exactly 2. Therefore the correct answer is **c**.

4 The recurrent 3's indicate that the difference between the limits is 3. This immediately eliminates choices **a** and **b**. Now that the 3's have been explained, the numerators of each term can be factored to $1\cdot 3$, $2\cdot 3$, $3\cdot 3$, etc., and the coefficients of 3 represent the index of summation (the "counting variable"). Then the terms of the sum represent x^2, so the answer is **d**.

5 The integral can be split up: $\int_{-3}^{3} f(x)dx + \int_{-3}^{3} 1dx$. The area of A cancels the area of B, so $\int_{-3}^{3} f(x)dx = -2$. Then $\int_{-3}^{3} 1dx$ simply represents the area of a rectangle with height 1 and width $3-(-3)=6$; this area is $1\cdot 6=6$. Therefore the total area is $-2+6=4$, choice **c**.

6a Make the substitution $u = 3+4\sin x$, so $du = 4\cos x\,dx$. The integral then becomes $\int u^{2005}du$. By the Power Rule for Antiderivatives, this is $\frac{1}{2006}u^{2006} + C$. Undoing the *u*-substitution, we have $\frac{1}{2006}(3+4\sin x)^{2006} + C$.

6b Make the substitution $u = \sqrt{3-x^2}$, so $du = -\frac{x}{\sqrt{3-x^2}}dx$. The integral then becomes $\int -3^u\,du = \frac{3^u}{\ln 3} + C$. Reversing the substitution gives $-\frac{3^{\sqrt{3-x^2}}}{\ln 3} + C$.

6c Make the substitution $u = \sin^{-1} x$, so $du = \frac{dx}{\sqrt{1-x^2}}$. The integral then becomes $\int u\,du = \frac{1}{2}u^2 + C$. Reversing the substitution gives $\frac{1}{2}\left(\sin^{-1} x\right)^2 + C$.

6d Break the integral up into $\int_{-3}^{3} 4dx + \int_{-3}^{3} \sqrt{9-x^2}\,dx$. The first integral is the area of a rectangle with height 4 and width 6, which is 24, and the second is half the area of a circle of radius 3, which is $\frac{1}{2}\left(3^2\pi\right) = \frac{9}{2}\pi$. Therefore the sum is $24 + \frac{9}{2}\pi$.

6e Make the substitution $u = \ln x$, so $du = \frac{dx}{x}$. The integral then becomes $\int_{\ln(1)}^{\ln(e)} \sin u\,du = \int_{0}^{1} \sin u\,du$. Using the Fundamental Theorem of Calculus, Part Two, this is $-\cos u\big]_0^1 = -\cos 1 - (-\cos 0) = -\cos 1 + 1$.

7a $g(4)$ is represented by the area under the graph of *f* from 0 to 4. The area from 0 to 2 cancels to 0, and the rest is a trapezoid with area 3. $g'(4)$ is represented by $f(4)$ per the Fundamental Theorem of Calculus, Part One, and $f(4)=0$. Finally, $g''(4) = f'(4) = -2$.

7b Since $g'(x) = f(x)$ and $f(x)$ changes sign at $x=1$, there is a relative extreme at $x=1$. Specifically, $f(x)$ goes from negative to positive, this is a local minimum.

7ci Each period adds an area of 2, because the rest of the area cancels (negative and positive signed area). Going from 0 to 10 is two periods, so $g(10) = \int_0^{10} f(x)dx = 2\cdot 2 = 4$.

7cii The slope at $x=108$ is $g'(108) = f(108) = f(3)$ since *f* has period 5, and $f(3)=2$. Now the *y*-coordinate of $g(108)$ is determined by the number of periods *f* has gone through at 108. This is 20 full periods (for an area of 40), and additionally it's gone through the first $\frac{3}{5}$ of the next period. This adds another area of 2, for a total area of 42, and a *y*-coordinate of 42. Thus the line has equation $y-42 = 2(x-108)$.

8a The 20 indicates that there are 20 rectangles in the approximation. The 2 indicates that the lower limit is 2, and the 5 means that the interval's length is $\frac{20}{5}=4$. Therefore the upper limit is 6. The quantity inside the parentheses means that the integrand is x^2. Therefore the Riemann sum is an approximation to $\int_2^6 x^2\,dx$ using 20 right rectangles.

8b Recall that a right rectangle approximation to $\int_a^b f(x)dx$ takes the form $\frac{b-a}{n}\sum_{i=1}^{n} f\left(a+\frac{i(b-a)}{n}\right)$. Plugging in the relevant pieces gives $\frac{4}{n}\sum_{i=1}^{n}\left(2+\frac{4i}{n}\right)^2$.

8c As discussed, this is $\int_2^6 x^2\,dx$.

9 Make the substitution: $u=2x+1$ gives $du=2dx$. Then the integral is $\int_{2\cdot 0+1}^{2\cdot 2+1}\frac{1}{2}\sqrt{u}\,du=\frac{1}{2}\int_1^5\sqrt{u}\,du$. This is choice **c**.

10 This requires use of the Fundamental Theorem of Calculus, Part One, and the Chain Rule. It would be $\sin\left(\left(x^2\right)^3\right)=\sin\left(x^6\right)$, but we also have to invoke the Chain Rule: the derivative of the upper limit, x^2, is $2x$, by which we multiply $\sin\left(x^6\right)$ to get $2x\sin\left(x^6\right)$, choice **e**.

11 Make the substitution $u=x^3$, so $du=3x^2\,dx$. The integral then becomes $\int\frac{1}{3}\cos u\,du=\frac{1}{3}\sin u+C=\frac{1}{3}\sin\left(x^3\right)+C$, choice **b**.