AP Calculus BC
Review — Applications of Integration (Chapter 6)

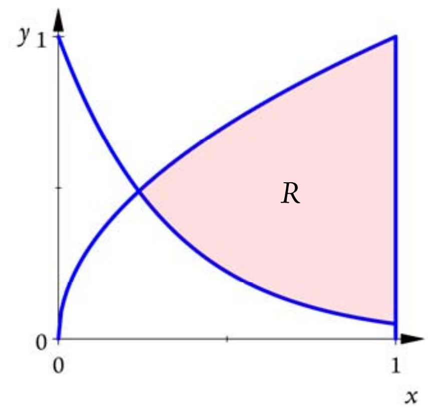
Things to Know and Be Able to Do

- Find the area between two curves by integrating with respect to x or y
- Find volumes by approximations with cross sections: disks (cylinders), washers, and other shapes
- Find volume by cylindrical shells: (radius r, height h, and thickness dr gives volume $dV = 2\pi rh\,dr$)
- Find work done using the formula $W = \int F\,dx$, noting that one common instance of a force is weight

Practice Problems

For all problems, show a correct, labeled diagram and a complete setup of the problem in terms of a single variable. Use correct units where applicable. This is designed to be done with a calculator. Remember, when giving approximate answers, to give three decimal places.

1 Let R be the shaded region bounded by the graphs of $y=\sqrt{x}$ and $y=e^{-3x}$ and the vertical line $x=1$, as shown in the figure at right.



a Find the area of R.

b Find the volume of the solid generated when R is revolved about the horizontal line $y=1$.

c The region R is the base of a solid. For this solid, each cross-section perpendicular to the x-axis is a rectangle whose height is 5 times the length of its base in region R. Find the volume of this solid.

2 A container is in the shape of a regular square pyramid. The height of the pyramid is 6 ft and the sides of the square base are 4 ft long. The tank is full of a liquid with weight density 68 lb/ft^3. Find the work done in pumping the liquid to a point 4 ft above the top of the tank.

3 Let R be the region in the first quadrant bounded by the graph of $y=x-x^3$ and the x-axis. Find the volume of the solid generated when R is revolved about the (**a**) x-axis and (**b**) y-axis

4 If the force F, in $\text{ft}\cdot\text{lb}$, acting on a particle on the x-axis is given by $F(x)=\frac{1}{x^2}$, then the work done in moving the particle from $x=1$ ft to $x=3$ ft is equal to

a $2 \text{ ft}\cdot\text{lb}$ **b** $\frac{2}{3} \text{ ft}\cdot\text{lb}$ **c** $\frac{26}{27} \text{ ft}\cdot\text{lb}$ **d** $1 \text{ ft}\cdot\text{lb}$ **e** $\frac{3}{2} \text{ ft}\cdot\text{lb}$

5 The base of a solid is a circle of radius a, and every plane cross-section perpendicular to one specific diameter is a square. The solid has volume

a $\frac{8}{3}a^3$ **b** $2\pi a^3$ **c** $4\pi a^3$ **d** $\frac{16}{3}a^3$ **e** $\frac{8\pi}{3}a^3$

6 The region whose boundaries are $y=3x-x^2$ and $y=0$ is revolved about the x-axis. The resulting solid has volume

a $\pi\int_0^3 \left(9x^2+x^4\right)dx$ **b** $\pi\int_0^3 \left(3x-x^2\right)^2 dx$ **c** $\pi\int_0^{\sqrt{3}} \left(3x-x^2\right)dx$

d $2\pi\int_0^3 y\sqrt{9-4y}\,dy$ **e** $\pi\int_0^{9/4} y^2\,dy$

7 The area of the region enclosed by the graphs of $y = x^2$ and $y = x$ is

a $\frac{1}{6}$ **b** $\frac{1}{3}$ **c** $\frac{1}{2}$ **d** $\frac{5}{6}$ **e** 1

8 When the region enclosed by the graphs of $y = x$ and $y = 4x - x^2$ is revolved about the *y*-axis, the volume of the solid generated is given by

a $\pi\int_0^3 \left(x^3 - 3x^2\right)dx$ **b** $\pi\int_0^3 \left(x^3 - \left(4x - x^2\right)^2\right)dx$ **c** $\pi\int_0^3 \left(3x - x^2\right)^2 dx$

d $2\pi\int_0^3 \left(x^3 - 3x^2\right)dx$ **e** $2\pi\int_0^3 \left(3x^2 - x^3\right)dx$

9 What is the volume of the solid generated by rotating about the *x*-axis the region enclosed by the graph of $y = \sec x$ and the lines $x = 0,\ y = 0,$ and $x = \frac{\pi}{3}$?

a $\frac{\pi}{\sqrt{3}}$ **b** π **c** $\pi\sqrt{3}$ **d** $\frac{8\pi}{3}$ **e** $\pi\ln\left(\frac{1}{2} + \sqrt{3}\right)$

10 If the region in the first quadrant bounded between the *y*-axis and the graph of $x = 2y(3 - y)^2$ is revolved about the *x*-axis, the volume of the solid generated is given by

a $\int_0^3 \pi\left(2y\left(3 - y^2\right)\right)^2 dy$ **b** $\int_0^8 2\pi x\left(2x(3 - x)^2\right)dx$ **c** $\int_0^8 2\pi x\left(3 - \sqrt{\frac{x}{2}}\right)dx$

d $\int_0^3 4\pi y^2 (3 - y)^2 dy$ **e** $\int_0^8 \pi\left(2x(3 - x)^2\right)dx$

11 Find the area enclosed by the graphs of $y = x^3 + 2x^2 - 10x - 12$ and $y = x$.

a $\frac{343}{12}$ **b** $\frac{99}{4}$ **c** $\frac{160}{3}$ **d** $\frac{937}{12}$ **e** $\frac{385}{12}$

Answers

1a 0.443
1b 1.424
1c 1.554
2 18496 ft · lb

3a $\frac{8\pi}{106} \approx 0.239$
3b $\frac{4\pi}{15} \approx 0.838$

4 b
5 d
6 b
7 a

8 e
9 c
10 d
11 d

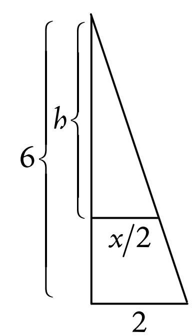
Solutions

1a First we need to find the beginning of the interval over which to integrate, which is the point of intersection of $y=\sqrt{x}$ with $y=e^{-3x}$. Therefore we solve $\sqrt{x}=e^{-3x}$; this cannot be solved for x exactly, but an approximation can be found: 0.239. Since the top function is $y=\sqrt{x}$, the bottom function is $y=e^{-3x}$, and the upper limit is 1, we integrate $\int_{0.239}^{1}\left(\sqrt{x}-e^{-3x}\right)dx$. This can be evaluated as $\frac{1}{3}e^{-3x}+\frac{2}{3}x^{3/2}\Big]_{0.239}^{1}$ or just plugged into a calculator; the answer is 0.443.

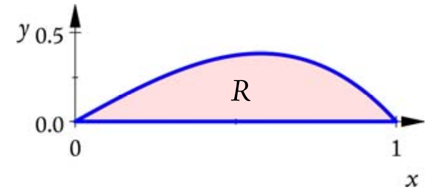
1b We find this object's volume using disks centered around the line $y=1$. Each disk has inner radius $1-\sqrt{x}$ and outer radius $1-e^{-3x}$, so each one has area $dA=\pi\left(\left(1-e^{-3x}\right)^2-\left(1-\sqrt{x}\right)^2\right)$ and, with thickness dx, volume $dV=\pi\left(\left(1-e^{-3x}\right)^2-\left(1-\sqrt{x}\right)^2\right)dx$. To find the total volume, we integrate $\int_{0.239}^{1}\pi\left(\left(1-e^{-3x}\right)^2-\left(1-\sqrt{x}\right)^2\right)dx$. Don't bother finding an antiderivative for the integrand; it's really ugly and you'll need to approximate the answer anyway. Your calculator will give you the approximation $V=1.424$.

1c Each rectangle has width $\sqrt{x}-e^{-3x}$ and height $5\left(\sqrt{x}-e^{-3x}\right)$. They each have area $dA=5\left(\sqrt{x}-e^{-3x}\right)^2$, and if their thickness is dx, each volume is $dV=5\left(\sqrt{x}-e^{-3x}\right)^2 dx$. The total volume is given by $V=\int_{0.239}^{1}5\left(\sqrt{x}-e^{-3x}\right)^2 dx$. An approximation to this is $V=1.554$.

2 Consider a square horizontal "slab" of liquid at a height h below the pyramid's apex and with side length x. A resulting side view of half the pyramid is shown at right. Clearly, the two triangles are similar, so we can set up the proportion $\frac{h}{x/2}=\frac{6}{2}$, meaning $x=\frac{2}{3}h$. Thus a slab located h below the apex has side length $\frac{2}{3}h$, area $dA=\left(\frac{2}{3}h\right)^2=\frac{4}{9}h^2$, and if it has thickness dh, volume $dV=\frac{4}{9}h^2\,dh$. This means that each slab's weight is $68\left(\frac{4}{9}h^2\,dh\right)=\frac{272}{9}h^2\,dh$. Each has to be lifted a distance h to get to the apex and then a further 4 to the desired point, for a total distance of $h+4$. Therefore the work done to lift each slab is $dW=\frac{272}{9}h^2\,dh\left(4+h\right)$, and the total work is $\int_0^6\frac{272}{9}h^2\left(4+h\right)dh=18496$ ft · lb.



3a A diagram of the region is shown at right. The volume can be found by disks centered around the x-axis; each disk has radius $y=x-x^3$ and thickness dx, for a volume of $dV=\pi\left(x-x^3\right)^2 dx$. The object's total volume is then given by $V=\int_0^1\pi\left(x-x^3\right)^2 dx=\frac{8\pi}{105}\approx 0.239$.

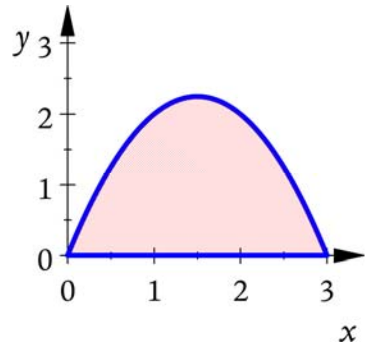


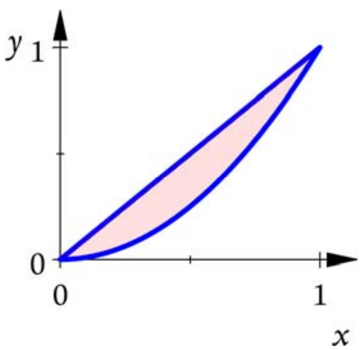
3b This requires the method of cylindrical shells, which should be centered around the *y*-axis. Each shell has radius *x*, thickness *dx*, and height $y = x - x^3$. So each cylinder has volume $dV = 2\pi x\left(x - x^3\right)dx$. The total volume is given by $V = \int_0^1 2\pi x\left(x - x^3\right)dx = \dfrac{4\pi}{15} \approx 0.838$.

4 Since $W = \int F\,dx$ and the particle is moving from $x = 1$ ft to $x = 3$ ft under a force of $F = \dfrac{1}{x^2}$ lb, the total work done is $\int_{1\text{ ft}}^{3\text{ ft}} \dfrac{1}{x^2}dx = -\dfrac{1}{x}\Big]_{1\text{ ft}}^{3\text{ ft}} = \dfrac{2}{3}$ ft · lb. This is choice **b.**

5 The circle is given by $x^2 + y^2 = a^2$, so $x^2 = a^2 - y^2$. Each square has base 2*x* and height 2*x* for an area of $(2x)^2 = 4x^2$. Since the squares are parallel to the *x*-axis, they have thickness *dy*, and each "slab" has volume $dV = 4x^2\,dy$. Fortunately, since we know $x^2 = a^2 - y^2$, we can substitute that in to find $dV = 4\left(a^2 - y^2\right)dy$. Then the total volume is $V = \int_{-a}^{a} 4\left(a^2 - y^2\right)dy = 4\left(a^2 y - \tfrac{1}{3}y^3\right)\Big]_{-a}^{a} = \tfrac{8}{3}a^3 - \left(-\tfrac{8}{3}a^3\right) = \tfrac{16}{3}a^3$, choice **d.**

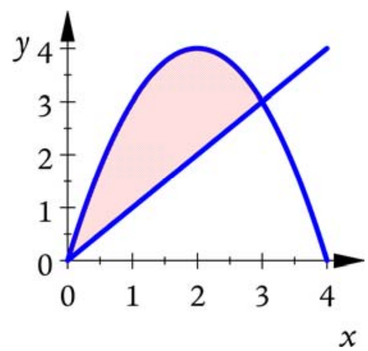
6 The region is shown at right; its left boundary is at $x = 0$ and its right boundary is at $x = 3$. The volume of the solid described is found by disks centered around the *x*-axis; each disk has radius $3x - x^2$. If the disks have thickness *dx*, each one's volume is $dV = \pi\left(3x - x^2\right)^2 dx$, so the total volume is given by $V = \int_0^3 \pi\left(3x - x^2\right)^2 dx$. This is choice **b.**

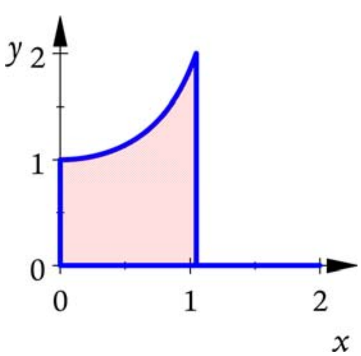




7 The region is shown at left; its right boundary is at $x = 1$; the "top" function is $y = x$, and the "bottom" function is $y = x^2$. Therefore, the region's area is $A = \int_0^1 \left(x - x^2\right)dx = \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\Big]_0^1 = \tfrac{1}{2}(1)^2 - \tfrac{1}{3}(1)^3 - \tfrac{1}{2}(0)^2 - \tfrac{1}{3}(0)^3 = \tfrac{1}{6}$, choice **a.**

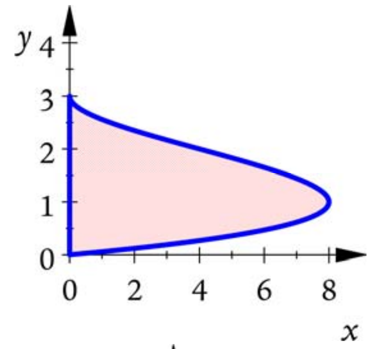
8 The region is shown at right; its left boundary is at $x = 0$ and its right boundary is at $x = 3$. We can find the volume of the solid described with cylindrical shells centered around the *y*-axis. Each shall has radius *x*, thickness *dx*, and height $\left(4x - x^2\right) - x = 3x - x^2$. Therefore the volume of the solid is $2\pi\int_0^3 \left(3x - x^2\right)x\,dx$, or $2\pi\int_0^3 \left(3x^2 - x^3\right)dx$, choice **e.**





9 The region is shown at left. We can find the volume of the solid with disks centered around the *x*-axis. Each disk has radius $y = \sec x$ and thickness *dx*, so its volume is $dV = \pi\sec^2 x\,dx$. The total volume is thus $\int_0^{\pi/3} \pi\sec^2 x\,dx = \pi\tan x\Big]_0^{\pi/3} = \pi\sqrt{3}$, choice **c.**

10 The region is shown at right. Its lower boundary is $y=0$ and its upper boundary is $y=3$. Finding the volume described is tricky; we need to use cylindrical shells centered around the x-axis. Each shell has radius y, thickness dy, and height $x=2y(3-y)^2$. The volume of each shell is given by $dV=2\pi y\left(2y(3-y)^2\right)dy$, so the total volume is $dV=\int_0^3 2\pi y\left(2y(3-y)^2\right)dy=4\pi\int_0^3 y^2(3-y)^2\,dy$, choice **d**.



11 The graphs with the *two* regions in question shown are at right. The left boundary of the left region is $x=-4$, the curves intersect at $x=-1$, and the right boundary of the right region is $x=3$. Since in the left region the "top" function is $y=x^3+2x^2-10x-12$ while in the right region the "top" function is $y=x$, the left region's area is $\int_{-4}^{-1}\left(\left(x^3+2x^2-10x-12\right)-x\right)dx=\int_{-4}^{-1}\left(x^3+2x^2-11x-12\right)dx$ and the right's is $\int_{-1}^{3}\left(x-\left(x^3+2x^2-10x-12\right)\right)dx=\int_{-1}^{3}\left(-x^3-2x^2+11x+12\right)dx$. The first integral is evaluated as $\left.\tfrac{1}{4}x^4+\tfrac{2}{3}x^3-\tfrac{11}{2}x^2-12x\right]_{-4}^{-1}=\tfrac{99}{4}$, and the second as $\left.-\tfrac{1}{4}x^4-\tfrac{2}{3}x^3+\tfrac{11}{2}x^2+12x\right]_{-1}^{3}=\tfrac{160}{3}$. The total area is thus $\tfrac{99}{4}+\tfrac{160}{3}=\tfrac{937}{12}$, choice **d.**