AP Calculus BC
Review — Inverse Functions (Chapter 7)

Things to Know and Be Able to Do

- How to find an inverse function's derivative at a particular point (page 418)
- The following derivatives (including, for the inverse trigonometric ones, how to derive them):
  - $\frac{d}{dx}\left(b^x\right) = b^x \ln b$ for $b > 0$
    - special case: $\frac{d}{dx}\left(e^x\right) = e^x$
  - $\frac{d}{dx}\left(\log_b |x|\right) = \frac{1}{x \ln b}$ for $b > 0$ and $b \neq 1$
    - special case: $\frac{d}{dx}\left(\ln|x|\right) = \frac{1}{x}$
  - $\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$
  - $\frac{d}{dx}\left(\csc^{-1} x\right) = -\frac{1}{x\sqrt{x^2-1}}$
  - $\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$
  - $\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}}$
  - $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$
  - $\frac{d}{dx}\left(\cot^{-1} x\right) = -\frac{1}{1+x^2}$
- The following antiderivatives and the all-new notation for them:
  - $\int b^x dx = \frac{b^x}{\ln b} + C$ for $b > 0$
    - special case: $\int e^x dx = e^x + C$
  - $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$
  - $\int -\frac{dx}{x\sqrt{x^2-1}} = \csc^{-1} x + C$
  - $\int -\frac{dx}{\sqrt{1-x^2}} = \cos^{-1} x + C$
  - $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C$
  - $\int \frac{dx}{1+x^2} = \tan^{-1} x + C$
  - $\int -\frac{dx}{1+x^2} = \cot^{-1} x + C$
- The shapes of the graphs of $y = \sin^{-1} x$, $y = \cos^{-1} x$, and $y = \tan^{-1} x$ along with each function's domain and range
- l'Hôpital's Rule and when it applies, including how to convert indeterminate forms of types $0 \cdot \infty$, $\infty - \infty$, 0^0, ∞^0, and 1^∞ into forms to which l'Hôpital's Rule applies

Practice Problems

All questions should be completed without the use of a calculator.

1 Find $\frac{dy}{dx}$ or the other specified derivative for each function given.

a $y=\tan^{-1}\sqrt{x-1}$

b $y=7\log_5\left(e^{2x}\right)$

c $y=\left(\sin^{-1}x\right)^4$

d Given $q=\sec^2\left(7^{-2\log_7 t}\right)$, find $\frac{dq}{dt}$

e Given $z=\frac{2}{x^3}-\frac{1}{x}+5^{x^3}+2^e+\ln 8$, find $\frac{dz}{dx}$

2 Find a rule for $f^{(n)}(x)$ if $f(x)=\ln(2x)$.

3 Find a general antiderivative for each function given.

a $f(x)=\frac{1}{x^3}-\frac{1}{x}+\ln(3)3^x+2^e+\ln 8$

b $f(x)=\frac{2x}{9+x^2}$

c $f(x)=\frac{1}{\sqrt{9-x^2}}$

d $f(x)=\frac{3x^2\ln 4\cdot 4^{\tan^{-1}x^3}}{1+x^6}$

4 Determine the function $f(x)$ if $f''(x)=(\ln 2)^2 2^x-\frac{1}{x^2}$ and the equation of the tangent line to the graph of *f* at $x=1$ is $y+2=(2\ln 2)(x-1)$. Show your work.

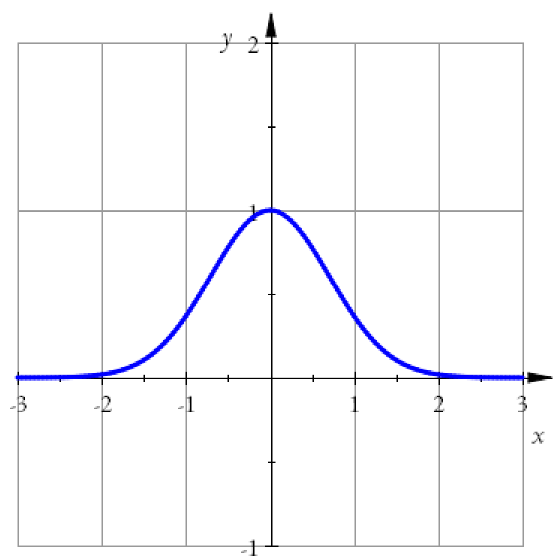
5 Evaluate each limit or show that it does not exist. Show your work.

a $\lim\limits_{x\to 0}\frac{\sin^2 x}{\cos(3x)-1}$

b $\lim\limits_{x\to 0}\frac{1-x^2-e^{-x^2}}{x^4}$

c $\lim\limits_{x\to 0}(1-3x)^{2/x}$

6 A rectangle has its base on the *x*-axis and two vertices on the curve $y=e^{-x^2}$, the graph of which is shown at right. Find the largest possible area for the rectangle. Justify your answer.



7 If $f(x)=\ln\left(x+4+e^{-3x}\right)$, then $f'(0)=$

a $-\frac{2}{5}$ **b** $\frac{1}{5}$ **c** $\frac{1}{4}$ **d** $\frac{2}{5}$ **e** nonexistent

8 Find the derivative of $y=3^{5^{x^2}}$ with respect to *x*.

a $2x\ln 15\cdot 15^{x^2}$

b $2x\ln 3\cdot 3^{5^{x^2}}$

c $2x\ln 3\cdot 5^{x^2}3^{5^{x^2}}$

d $2x\ln 5\cdot\ln 3\cdot 5^{x^2}3^{5^{x^2}}$

e none of these

9 Let *f* be the function defined by $f(x)=x^3+x$. If $g(x)=f^{-1}(x)$ and $g(2)=1$, find $g'(2)$.

a $\frac{1}{13}$ **b** $\frac{1}{4}$ **c** $\frac{7}{4}$ **d** 4 **e** 13

10 Which of the following is equivalent to $\sec\left(\cot^{-1}\frac{x}{3}\right)$?

a $\frac{3}{\sqrt{x^2+9}}$ **b** $\frac{x}{\sqrt{9-x^2}}$ **c** $\frac{\sqrt{x^2+9}}{3}$ **d** $\frac{\sqrt{x^2+9}}{x}$ **e** $\frac{\sqrt{9-x^2}}{x}$

11 Which of the following is the derivative of $f(x)=(\ln x)^x$ with respect to *x*?

a $x(\ln x)^{x-1}$

b $(\ln x)^{x-1}$

c $(\ln x)^x \ln(\ln x)$

d $(\ln x)^x\left(\ln(\ln x)+\frac{1}{\ln x}\right)$

e $\frac{x(\ln x)^{x+1}}{x+1}$

12 Find $\lim_{x\to 0^+} \arctan(\ln x)$

a 1 **b** ∞ **c** $\frac{\pi}{4}$ **d** $-\frac{\pi}{4}$ **e** $-\frac{\pi}{2}$

Answers

1a $\frac{dy}{dx} = \frac{1}{2x\sqrt{x-1}}$

1b $\frac{dy}{dx} = \frac{14}{\ln 5}$

1c $\frac{dy}{dx} = \frac{4\left(\sin^{-1} x\right)^3}{\sqrt{1-x^2}}$

1d $\frac{dq}{dt} = \frac{-4\sec^2\left(t^{-2}\right)\tan\left(t^{-2}\right)}{t^3}$

1e $\frac{dz}{dx} = -\frac{6}{x^4} + \frac{1}{x^2} + 3x^2 \ln 5 \cdot 5^{x^3}$

2 $f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{x^n}$

3a $F(x) = -\frac{1}{2x^2} - \ln|x| + 3^x + 2^e x + (\ln 8)x + C$

3b $F(x) = \ln\left(9 + x^2\right) + C$

3c $F(x) = \sin^{-1} \frac{x}{3} + C$

3d $F(x) = 4^{\tan^{-1}\left(x^3\right)} + C$

4 $f(x) = 2^x + \ln|x| - x - 3$

5a $-\frac{2}{9}$ **5b** $-\frac{1}{2}$ **5c** e^{-6}

6 $\sqrt{\frac{2}{e}}$

7 a **8** d **9** b **10** d **11** d **12** e

Solutions

1a Apply the Chain Rule to $(a \circ b)(x)$, letting $a(x) = \tan^{-1} x$ and $b(x) = \sqrt{x-1}$. Then $a'(x) = \frac{1}{x^2+1}$ and $b'(x) = \frac{1}{2\sqrt{x-1}}$, so $\frac{d}{dx}(a \circ b)(x) = \frac{1}{\sqrt{x-1}^2 + 1} \cdot \frac{1}{2\sqrt{x-1}} = \frac{1}{2x\sqrt{x-1}}$.

1b Simplify $7\log_5\left(e^{2x}\right) = \frac{7\ln\left(e^{2x}\right)}{\ln 5} = \frac{7 \cdot 2x}{\ln 5} = \frac{14}{\ln 5}x$, so $\frac{dy}{dx} = \frac{14}{\ln 5}$.

1c Apply the chain rule to $(a \circ b)(x)$, letting $a(x) = x^4$ and $b(x) = \sin^{-1} x$. Then $a'(x) = 4x^3$ and $b'(x) = \frac{1}{\sqrt{1-x^2}}$.

Therefore $\frac{d}{dx}(a \circ b)(x) = 4\left(\sin^{-1} x\right)^3 \cdot \frac{1}{\sqrt{1-x^2}} = \frac{4\left(\sin^{-1} x\right)^3}{\sqrt{1-x^2}}$.

1d Simplify $7^{-2\log_7 t} = \left(7^{\log_7 t}\right)^{-2} = t^{-2}$, meaning $q = \sec^2\left(t^{-2}\right)$. Then applying the Chain Rule gives $\frac{dq}{dt} = 2\sec\left(t^{-2}\right)\sec\left(t^{-2}\right)\tan\left(t^{-2}\right)\left(-2t^{-3}\right) = \frac{-4\sec^2\left(t^{-2}\right)\tan\left(t^{-2}\right)}{t^3}$.

1e Differentiating term by term with the Chain Rule applied to the third term and noting that the fourth and fifth terms are simply constants gives $\frac{dz}{dx} = -\frac{6}{x^4} + \frac{1}{x^2} + 3x^2 \ln 5 \cdot 5^{x^3}$.

a	$f^{(a)}(x)$
1	$\frac{1}{x}$
2	$\frac{-1}{x^2}$
3	$\frac{2}{x^3}$
4	$\frac{-6}{x^4}$
5	$\frac{24}{x^5}$

2 Consider the table at right. You're out of luck if you don't see the pattern, but it's not too difficult: $f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{x^n}$.

3a By a combination of inspection and application of the Power Rule for Antiderivatives, $F(x) = -\frac{1}{2x^2} - \ln|x| + 3^x + 2^e x + (\ln 8)x + C$.

3b This involves reversing the Chain Rule; the $2x$ is the result of an x^2 term. $F(x) = \ln\left(9 + x^2\right) + C$.

3c This, too, involves reversing the Chain Rule. The function is equal to $\dfrac{1}{3\sqrt{1-\left(\frac{x}{3}\right)^2}}$, meaning that $F(x)=\sin^{-1}\left(\frac{x}{3}\right)+C$.

3d This is pretty ugly, but you're helped by the fact that a function and a bunch of its derivatives appear in the original function. It's admittedly tricky to see that $F(x)=4^{\tan^{-1}\left(x^3\right)}+C$.

4 The equation of the tangent line means that the point $(x,y)=(1,-2)$ is on the graph and that $f'(1)=2\ln 2$. Antidifferentiating $f''(x)$ gives $f'(x)=(\ln 2)2^x+\dfrac{1}{x}+C$; knowing that $f'(1)=2\ln 2$ means that we can solve $2\ln 2=(\ln 2)2^1+\dfrac{1}{1}+C\Rightarrow C=-1$, and therefore $f'(x)=(\ln 2)2^x+\frac{1}{x}-1$. Antidifferentiating that gives $f(x)=2^x+\ln|x|-x+C$, and since $f(1)=-2$, we know that $f(x)=2^x+\ln|x|-x-3$.

5a This limit is an indeterminate form of type $\dfrac{0}{0}$. Differentiating the top and bottom (separately) gives $\lim\limits_{x\to 0}\dfrac{2\sin x\cos x}{-3\sin 3x}$, which is itself an indeterminate form, also of type $\dfrac{0}{0}$. Therefore we must again apply l'Hôpital's Rule to get $\lim\limits_{x\to 0}\dfrac{-2\sin^2 x+2\cos^2 x}{-9\cos 3x}$. Now we can plug in $x=0$ to get $-\frac{2}{9}$.

5b This is also an indeterminate form of type $\dfrac{0}{0}$. Applying l'Hôpital's Rule gives $\lim\limits_{x\to 0}\dfrac{-2x+2xe^{-x^2}}{4x^3}$, from which we can factor out and cancel an x to get $\lim\limits_{x\to 0}\dfrac{-2+2e^{-x^2}}{4x^2}$. This is also an indeterminate form of type $\dfrac{0}{0}$, so we apply l'Hôpital's Rule once again to get $\lim\limits_{x\to 0}\dfrac{-4xe^{-x^2}}{8x}$. An x cancels, so the limit is $\lim\limits_{x\to 0}\dfrac{-4e^{-x^2}}{8}$, into which we can plug $x=0$ to get $-\frac{1}{2}$.

5c This indeterminate form is of type 1^∞, so we cannot directly apply l'Hôpital's Rule. Therefore we rewrite it as $e^{\lim\limits_{x\to 0}\ln(1-3x)^{2/x}}=e^{\lim\limits_{x\to 0}\frac{2\ln(1-3x)}{x}}$. The limit in the exponent is an indeterminate form of type $\dfrac{0}{0}$, and applying l'Hôpital's Rule gives $e^{\lim\limits_{x\to 0}\frac{\frac{2(-3)}{1-3x}}{1}}=e^{\lim\limits_{x\to 0}\frac{-6}{1-3x}}$. We can now plug in $x=0$ to get e^{-6}.

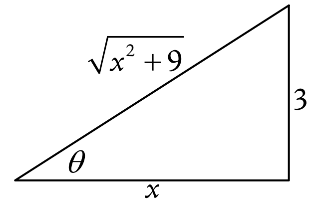
6 Let x be the x-coordinate of the rectangle's vertices in the first quadrant; therefore, $-x$ is the x-coordinate of the rectangle's vertices in the second quadrant. The rectangle's area is $A=2xe^{-x^2}$, so $\frac{dA}{dx}=2e^{-x^2}\left(1-2x^2\right)$. We wish to maximize this, so we set $2e^{-x^2}\left(1-2x^2\right)=0$. Then the Zero Product Property implies $2e^{-x^2}=0$ and/or $1-2x^2=0$, but the former will never be true. Therefore $2x^2=1\Rightarrow x=\dfrac{1}{\sqrt{2}}$. Given that, $A\left(\frac{1}{\sqrt{2}}\right)=\sqrt{\dfrac{2}{e}}$. We do, however, need to verify that this is a relative maximum. Thus we find $\frac{d^2A}{dx^2}=2e^{-x^2}(-4x)+\left(1-2x^2\right)\left(-4xe^{-x^2}\right)$ $=-4xe^{-x^2}\left(3-2x^2\right)$. Plugging in $x=\frac{1}{\sqrt{2}}$ gives $\left.\frac{d^2A}{dx^2}\right|_{x=\frac{1}{\sqrt{2}}}=-4\left(\frac{1}{\sqrt{2}}\right)e^{-1/2}\left(3-2\left(\frac{1}{2}\right)\right)=-\frac{4}{\sqrt{2}}e^{-1/2}\cdot 2$. This is negative, so the point is indeed a relative maximum.

7 $f'(x)=\dfrac{1}{x+4+e^{-3x}}\left(1-3e^{-3x}\right)$, so plugging in $x=0$ gives $f'(0)=\frac{1}{5}(1-3)=-\frac{2}{5}$, choice **a**.

8 Consider $f = 3^g$, $g = 5^h$, and $h = x^2$. Then $y = f\left(g\left(h(x)\right)\right)$. $\frac{df}{dg} = 3^g \ln 3 = 3^{5^h} \ln 3 = 3^{5^{x^2}} \ln 3$, $\frac{dg}{dh} = 5^h \ln 5 = 5^{x^2} \ln 5$, and $\frac{dh}{dx} = 2x$. Since $\frac{dy}{dx} = \frac{df}{dg}\frac{dg}{dh}\frac{dh}{dx}$ by the Chain Rule, $\frac{dy}{dx} = 3^{5^{x^2}} \ln 3 \cdot 5^{x^2} \ln 5 \cdot 2x$, which rearranges to choice **d**.

9 Clearly $f'(x) = 3x^2 + 1$ and $f'(1) = 4$, and Theorem 7 from section 7.1 (page 418) gives $g'(a) = \frac{1}{f'\left(g(a)\right)}$ for this situation where $a = 2$. Given that $g(2) = 1$, we know $g'(2) = \frac{1}{f'(1)} = \frac{1}{4}$, choice **b**.

10 Consider the diagram at right, in which $\theta = \cot^{-1} \frac{x}{3}$. In the diagram, $\sec\theta = \frac{\sqrt{x^2+9}}{x}$, choice **d**.



11 The function can be rewritten as $f(x) = e^{x\ln(\ln x)}$, and judicious application of the Chain and Product Rules gives $f'(x) = (\ln x)^x \left(\ln(\ln x) + \frac{1}{\ln x} \right)$.

12 Since $\lim_{x \to 0^+} \ln x = -\infty$, and $\lim_{x \to -\infty} \arctan x = -\frac{\pi}{2}$, the limit is $-\frac{\pi}{2}$, choice **e**.