AP Calculus BC
Review — Chapter 8, Part 2, and Chapter 9

Things to Know and Be Able to Do

- Know everything from the first part of Chapter 8
- Given an integrand, figure out how to antidifferentiate it using any of the following techniques or combinations thereof: the Power Rule for Antiderivatives, recognition, *u*-substitution, integration by parts, substitution with Pythagorean identities, trigonometric substitution, partial fractions, and other methods as specified in class
- Approximate integrals with left, right, midpoint, and trapezoidal Riemann sums (not new material) and Simpson's Rule
- Given an error bound formula for any of the above approximations, use it to find necessary numbers of subintervals, etc.; you need not memorize these formulae, but be sure to understand what each part means. The formulas are given on pages 557 and 561.
- Understand the two principal types of improper integrals and how to determine whether each converges; if an integral converges, know how to evaluate it in terms of limits (if possible) and how to determine the error if an approximation is made to the improper integral with an infinite limit.
- Find the arc length of explicitly-defined curves (both y in terms of x and x in terms of y) using the formulas, for endpoints a and b, $s_{a,b}=\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ or $s_{a,b}=\int_a^b \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$. Understand where these formulas came from (pages 583–585).
- Find the surface area of a surface of revolution: if the surface is given by a function rotated about the x-axis, its area is $S=\int 2\pi y\,ds$ where ds is defined by arc length, as above, or if the function is rotated about the y-axis, its area is $S=\int 2\pi x\,ds$. Again, understand where these formulas came from (pages 591–593).

Practice Problems

These problems may be done with a calculator except where noted otherwise. The original test, of course, required that you show all calculations.

1 The rate at which water flows out of a pipe, in gallons per hour, is given by a differentiable function R of time t. The table below shows the rates as measured every 3 hours for a 24-hour period.

t [hr]	0	3	6	9	12	15	18	21	24
$R(t)$ [gal/hr]	9.6	10.4	10.8	11.2	11.4	11.3	10.7	10.2	9.6

a Use a midpoint Riemann sum with 4 subintervals of equal length to approximate $\int_{0\text{ hr}}^{24\text{ hr}} R(t)\,dt$. Using correct units, explain the meaning of your answer in terms of water flow.

b Use Simpson's Rule with 8 subintervals of equal length to approximate $\int_{0\text{ hr}}^{24\text{ hr}} R(t)\,dt$.

c Use the Trapezoid Rule with 4 subdivisions of equal length to approximate $\int_{0\text{ hr}}^{24\text{ hr}} R(t)\,dt$.

2 Find the arc length along the graph of $y^3=8x^2$ from $(x,y)=(1,2)$ to $(x,y)=(8,8)$. Show the complete setup; you may use your calculator to evaluate the integral.

3 Find the area of the surface of revolution that is generated by revolving the curve $y=e^x$ for $0\le x\le 1$ about the y-axis. Show the complete setup; you may use your calculator to evaluate the integral.

4 Let I be defined by $I=\int_1^\infty \frac{3+2\sin x}{x^3}dx$.

a Show, using non-calculator work, that I converges.

b Find a value of a for which $I\approx\int_1^a \frac{3+2\sin x}{x^3}dx$ has an error of less than 0.001. Justify your answer using non-calculator work.

5 Let I be defined by $I=\int_1^5 \ln x\,dx$.

a Estimate a midpoint sum for I using 2 subintervals of equal length.

b Determine the number n of subintervals of equal length needed so that the error bound for a midpoint sum M_n, given by $\left|I-M_n\right|\le\frac{K(b-a)^2}{24n^2}$, is less than or equal to 0.001. (Let K be an upper bound on $\left|f''(x)\right|$ on the given interval.)

6 Let f be the function defined by $f(x)=-\ln x$ on $0\le x\le 1$, and let R be the region between the graph of $y=f(x)$ and the x-axis. Find the volume of the solid generated by revolving R about the y-axis, or show that it is infinite. Justify your answer.

7 Evaluate $\int_3^\infty \frac{dx}{x^2-4}$, showing non-calculator work.

8 If $f(x)$ is positive and $f''(x)>0$ on the interval $a\le x\le b$, which of the following approximations to $\int_a^b f(x)dx$ must be too large?

I the right rectangle sum
II the midpoint sum
III the trapezoidal sum

a I only **b** II only **c** I and II only **d** III only **e** II and III only

9 If 3 equal subdivisions of $[-4,2]$ are used, what is the trapezoidal approximation of $\int_{-4}^2 \frac{e^{-x}}{2}dx$?

a $e^2+e^0+e^{-2}$ **b** $e^4+e^2+e^0$ **c** $e^4+2e^2+2e^0+e^{-2}$
d $\frac{1}{2}\left(e^4+e^2+e^0+e^{-2}\right)$ **e** $\frac{1}{2}\left(e^4+2e^2+2e^0+e^{-2}\right)$

10 The expression $\frac{1}{50}\left(\sqrt{\frac{1}{50}}+\sqrt{\frac{2}{50}}+\sqrt{\frac{3}{50}}+\text{L}+\sqrt{\frac{50}{50}}\right)$ is a Riemann sum approximation for

a $\int_0^1\sqrt{\frac{x}{50}}dx$ **b** $\int_0^1\sqrt{x}\,dx$ **c** $\frac{1}{50}\int_0^1\sqrt{\frac{x}{50}}dx$ **d** $\frac{1}{50}\int_0^1\sqrt{x}\,dx$ **e** $\frac{1}{50}\int_0^{50}\sqrt{x}\,dx$

11 The length of a curve from $x=1$ to $x=4$ is given by $\int_1^4\sqrt{1+9x^4}\,dx$. If the curve contains the point $(1,6)$, which of the following could be an equation for this curve?

a $y=3+3x^2$ **b** $y=5+x^3$ **c** $y=6+x^3$

d $y=6-x^3$ **e** $y=\frac{16}{5}+x+\frac{9}{5}x^5$

Answers

1a 258.6 gal
1b 257.2 gal
1c 255 gal
2 $\dfrac{104\sqrt{13}-125}{27} \approx 9.258$
3 7.055

4a see solutions
4b $a = 50$
5a $6\ln 2 \approx 4.159$
5b $n = 26$
6 $V = \dfrac{\pi}{2}$

7 $\dfrac{1}{4}\ln 5$
8 d **9** e **10** b **11** b

Solutions

1a Firstly, any integral of a rate gives a total amount. Therefore this integral gives an approximation to the total amount of water that flowed out of the pipe from $t = 0$ hr to $t = 24$ hr. To find the approximation to the integral, note that each subinterval has length 6 hr because the interval's total length is 24 hr and we are dividing it into 4 parts; then $\dfrac{24 \text{ hr}}{4} = 6$ hr. Therefore our subintervals, if they are to be of equal length, are $t \in (0 \text{ hr}, 6 \text{ hr})$, $t \in (6 \text{ hr}, 12 \text{ hr})$, $t \in (12 \text{ hr}, 18 \text{ hr})$, and $t \in (18 \text{ hr}, 24 \text{ hr})$. Then, because we are using the midpoint rule, our sample points are $t = 3$ hr, $t = 9$ hr, $t = 15$ hr, and $t = 21$ hr. So the 4-subinterval midpoint approximation to the integral is $\int_{0 \text{ hr}}^{24 \text{ hr}} R(t)\,dt \approx (6 \text{ hr})(10.4 \text{ gal/hr} + 11.2 \text{ gal/hr} + 11.3 \text{ gal/hr} + 10.2 \text{ gal/hr}) = 258.6$ gal.

1b This time, our subintervals have length 3 hr because $\dfrac{24 \text{ hr}}{8} = 3$ hr. The sample points are all the given values of *t*. A formula for Simpson's Rule is given on the top of page 560; using this, the approximation to the integral is

$$\int_{0 \text{ hr}}^{24 \text{ hr}} R(t)\,dt \approx \frac{1}{3}(3 \text{ hr})\big(R(0) + 4R(3) + 2R(6) + 4R(9) + 2R(12) + 4R(15) + 2R(18) + 4R(21) + R(24)\big)$$

$$= (1 \text{ hr})(9.4 + 4 \cdot 10.4 + 2 \cdot 10.8 + 4 \cdot 11.2 + 2 \cdot 11.4 + 4 \cdot 11.3 + 2 \cdot 10.7 + 4 \cdot 10.2 + 9.6) = 257.2 \text{ gal.}$$

1c We are back to subintervals of length 6 hr as in part **1a**. However, the sample points are now the endpoints of the subintervals, with all but the first and last ones counted twice each, and the total divided by two (averaging the two bases of each trapezoid). Thus the trapezoidal approximation to the integral in question is given by $\int_{0 \text{ hr}}^{24 \text{ hr}} R(t)\,dt \approx \frac{1}{2}(6)\big(R(0) + 2R(6) + 2R(12) + 2R(18) + R(24)\big) = \frac{1}{2}(6)(9.6 + 2 \cdot 10.8 + 2 \cdot 11.4 + 2 \cdot 10.7 + 9.6)$ $= 255$ gal.

2 First, solve the equation for *y* explicitly in terms of *x*: $y = 2x^{2/3}$. Therefore $\dfrac{dy}{dx} = \dfrac{4}{3}x^{-1/3}$ by the Power Rule. Since the arc length formula from *a* to *b* is $s_{a,b} = \int_a^b \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}\,dx$, we have $s_{1,8} = \int_1^8 \sqrt{1 + \dfrac{16}{9}x^{-2/3}}\,dx$. This is thoroughly unpleasant to evaluate by hand, but any CAS will gladly tell you that it's equal to $\dfrac{104\sqrt{13}-125}{27}$, and you can also approximate it with 9.258.

3 This is best done in terms of *x*, where $x = \ln y$ and $x' = \dfrac{1}{y}$. The limits will be $\left[e^0, e^1\right] = [1, e]$. Then the area is given by $S = 2\pi \int_1^e \ln y \sqrt{1 + \left(\dfrac{1}{y}\right)^2}\,dy$. The antiderivative of this is not an elementary function, but can be approximated; a CAS gives an approximation to the integral as 7.055.

4a We show this using the Comparison Test; that is, we must find a function f such that $\dfrac{3+2\sin x}{x^3} \le f(x)$ for $x \in [1,\infty)$ and $\int_1^\infty f(x)\,dx$ can be shown to converge. The numerator of the original integrand, $3+2\sin x$, ranges in value from 1 to 5 since $2\sin x$ ranges from -2 to 2; therefore, $\dfrac{5}{x^3} \ge \dfrac{3+2\sin x}{x^3}$ always, and it is fairly convenient to show that $\int_1^\infty \dfrac{5}{x^3}dx$ converges: it is equal to $\lim_{k\to\infty}\int_1^k \dfrac{5}{x^3}dx$. This integral simplifies to $\lim_{k\to\infty}\left(-\dfrac{5}{2}x^{-2}\Big]_1^k\right)$ $= \lim_{k\to\infty}\left(-\dfrac{5}{2}k^{-2}+\dfrac{5}{2}\right) = \dfrac{5}{2}$, which is finite. Therefore $\int_1^\infty \dfrac{3+2\sin x}{x^3}dx$ converges.

4b We can divide $\int_1^\infty \dfrac{3+2\sin x}{x^3}dx$ into two parts: $\int_1^a \dfrac{3+2\sin x}{x^3}dx + \int_a^\infty \dfrac{3+2\sin x}{x^3}dx$, in which the second term represents the error of the approximation. Thus we need to find a such that $\int_a^\infty \dfrac{3+2\sin x}{x^3}dx \le 0.001$. However, an antiderivative of $\dfrac{3+2\sin x}{x^3}$ cannot be expressed in terms of elementary functions, so we instead consider $\int_a^\infty \dfrac{5}{x^3}dx \le 0.001$, which is valid because the latter is always greater, meaning the error in approximating $\int_1^\infty \dfrac{5}{x^3}dx$ will always be *greater* than the error in approximating the integral in which we are actually interested. Since $\int_a^\infty \dfrac{5}{x^3}dx = \lim_{k\to\infty} -\dfrac{5}{2}x^{-2}\Big]_a^k = \lim_{k\to\infty}\left(-\dfrac{5}{2}k^{-2}+\dfrac{5}{2}a^{-2}\right) = \dfrac{5}{2}a^{-2}$, we are faced with the rather simple task of finding a such that $\dfrac{5}{2}a^{-2} \le 0.001$. Now all we need to do is divide both sides by $\dfrac{5}{2}$, giving $a^{-2} \le 0.0004$, and then take the reciprocal of the square root of both sides. Therefore, $a = 50$.

5a Each subinterval has length 2; the subintervals are $(1,3)$ and $(3,5)$, so the sample points are $x=2$ and $x=4$. Thus the approximation is $\int_1^5 \ln x\,dx \approx 2(\ln 2+\ln 4) = 2(\ln 2+\ln(2\cdot 2)) = 2(\ln 2+\ln 2+\ln 2) = 6\ln 2 \approx 4.159$.

5b The integrand's second derivative is $-\dfrac{1}{x^2}$, and an upper bound for its absolute value on the interval $[1,5]$ is $|f''(1)| = |-1| = 1$. Therefore we are interested in finding n given $0.001 \ge \dfrac{1(5-1)^2}{24n^2}$. The right side simplifies: $0.001 \ge \dfrac{2}{3}n^{-2}$, or $0.0015 \ge n^{-2}$; therefore, $0.0015n^2 \ge 1$ or $n^2 \ge 666.\overline{6}$, giving $n \ge 25.820$. Naturally, we want an integer value of n, so $n = 26$.

6 This must be done with cylindrical shells. Each one has $dV = 2\pi rh\,dr = 2\pi x(-\ln x)dx$, so $V = 2\pi\int_0^1 x(-\ln x)dx$. For simplicity's sake, we will momentarily ignore the factor of 2π. To find an antiderivative for this, use integration by parts with $u = \ln x$ and $dv = -x\,dx$. This gives $du = \dfrac{dx}{x}$ and $v = -\dfrac{1}{2}x^2$. Therefore $uv - \int v\,du = -\tfrac{1}{2}x^2\ln x + \int \tfrac{1}{2}x^2\dfrac{dx}{x} = -\tfrac{1}{2}x^2\ln x + \int \tfrac{1}{2}x\,dx = -\tfrac{1}{2}x^2\ln x + \tfrac{1}{4}x^2 + C$. Evaluating this at 1 is easy— it gives $\tfrac{1}{4}$— but at 0, we have an asymptote. Therefore we must concern ourselves with $\lim_{x\to 0}\left(-\tfrac{1}{2}x^2\ln x + \tfrac{1}{4}x^2\right)$. The second term clearly limits to zero, and the first term may be analyzed with L'Hôpital's Rule. We first rewrite it as $\dfrac{1}{2}\lim_{x\to 0}\dfrac{-\ln x}{x^{-2}}$. Now we have an indeterminate form of type $\dfrac{\infty}{\infty}$. Differentiating the numerator and denominator gives $\frac{1}{2}\lim_{x\to 0}\frac{-1/x}{-2/x^3}=\frac{1}{2}\lim_{x\to 0}\left(\frac{1}{2}x^2\right)$. Clearly, this is also zero. Therefore, evaluating the integral has given us $\frac{1}{4}-0=\frac{1}{4}$, which we must multiply by 2π to give $\frac{\pi}{2}$.

7 The integrand's denominator factors to $(x+2)(x-2)$, and integration by partial fractions of $\int\frac{dx}{(x+2)(x-2)}$ is executed as follows: we expect something of the form $\frac{A}{x+2}+\frac{B}{x-2}=\frac{1}{(x+2)(x-2)}$, so we know from clearing the denominators that $A(x-2)+B(x+2)=1$. This is rearranged to give $x(A+B)-2A+2B=1$, which allows us to set up the system $\left\{\begin{matrix} A+B=0 \\ -2A+2B=1 \end{matrix}\right\}$. This solves to $(A,B)=\left(-\frac{1}{4},\frac{1}{4}\right)$. Thus, the original integrand is equivalent to $-\frac{1/4}{x+2}+\frac{1/4}{x-2}$. Now we can integrate each term, giving $-\frac{1}{4}\ln(x+2)+\frac{1}{4}\ln(x-2)$, which can be rewritten as $\frac{1}{4}\ln\frac{x-2}{x+2}$. Now we evaluate $\frac{1}{4}\lim_{k\to\infty}\ln\frac{x-2}{x+2}\Big]_{x=3}^{x=k}$ as follows (ignoring the $\frac{1}{4}$; we'll put that back in at the end): $\lim_{k\to\infty}\left(\ln\frac{k-2}{k+2}-\ln\frac{3-2}{3+2}\right)=\lim_{k\to\infty}\left(\ln\frac{k-2}{k+2}-\ln\frac{1}{5}\right)=\ln 5+\lim_{k\to\infty}\ln\frac{k-2}{k+2}$. Since the function $f(k)=\ln\frac{k-2}{k+2}$ is continuous for very large values of k, we can pull out the natural logarithm function from the limit: $\ln 5+\ln\left(\lim_{k\to\infty}\frac{k-2}{k+2}\right)=\ln 5+\ln 1=\ln 5$. Now we need to return the $\frac{1}{4}$ to its rightful place; the answer is $\frac{1}{4}\ln 5$.

8 The fact that $f''(x)>0$ on the interval in question means that the function is concave up. Therefore a secant line, such as that used in the trapezoid sum, will always be above the function itself, making the trapezoid approximation too large. Therefore **III** must be one of the correct answers; options **a**, **b**, and **c** are incorrect. Without knowing anything more about the shape of the function, it is impossible to conclude whether a tangent line to the midpoint of a subinterval will lie entirely above or below the function or neither, so we cannot make a conclusion about the accuracy of the midpoint sum. Therefore **d** is the correct answer.

9 The subintervals are $[-4,-2]$, $[-2,0]$, and $[0,2]$; each has length 2. With the trapezoidal approximation, we count each subinterval's endpoint twice except for the first and last one's outer points, and divide by 2 on the outside to indicate averaging each trapezoid's bases. Thus the integral's trapezoidal approximation is given by $\frac{1}{2}(2)\left(\frac{e^{-4}}{2}+\frac{2e^{-2}}{2}+\frac{2e^{0}}{2}+\frac{e^{2}}{2}\right)=\frac{1}{2}\cancel{(2)\left(\frac{1}{2}\right)}\left(e^{-4}+2e^{-2}+2e^{0}+e^{2}\right)=\frac{1}{2}\left(e^{-4}+2e^{-2}+2e^{0}+e^{2}\right)$, choice **e**.

10 This can be rewritten as $\frac{1}{50}\sum_{k=1}^{50}\sqrt{\frac{k}{50}}$. The division by 50 indicates that there are 50 subintervals, and the fact that the coefficient's numerator is 1 indicates that the difference of the limits is 1. Remember that the summand takes the form $f\left(a+\frac{k(b-a)}{n}\right)$, which tells us that the lower limit of the integral is 0. Since we know that the difference of the limits is 1, the upper limit is 1, and since $n=50$, the function is $f(x)=\sqrt{x}$. This means that the correct response is **b**.

11 The arc length formula for y on $[a,b]$ is $s_{a,b}=\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$. Here, $\left(\frac{dy}{dx}\right)^2=9x^4$, so $\frac{dy}{dx}=3x^2$. This means that $y=x^3+C$ for some C; since $(x,y)=(1,6)$ is on the curve, **c** is not a possible answer and only **b** is correct.