Multivariable Calculus
Review Problems — Calculus BC Review and Hyperbolic Functions

Things to Know and Be Able to Do

- Understand hyperbolic trigonometric functions from definitions in terms of exponential functions and identities that relate them to each other, and notice parallels and differences between hyperbolic and circular trigonometric functions
- Prove the existence and value of limits of functions of one variable where they exist
- Recall concepts about integrals and derivatives from Calculus BC and these concepts' applications
- Use parametric equations to define and analyze curves in two dimensions
- Understand and use the two Theorems of Pappus that deal with solids of revolution
- Find the moments about axes and the center of mass of planar laminas of uniform density and of density given by a function of one variable, and of systems of particles

Practice Problems

These problems may be done with a calculator except where noted otherwise. The original test, of course, required that you show relevant work.

1a Show without the use of a calculator that $\int \frac{\cosh x}{\cosh^2 x - 1} dx = -\operatorname{csch} x + C$ for some constant C.

1b Use the result from part **a** to evaluate $\int_{\ln 2}^{\ln 5} \frac{\cosh x}{\cosh^2 x - 1} dx$ without the use of a calculator.

2 A system has three masses located at $(1,5)$, $(3,-2)$, and $(-2,-1)$. Their masses are 6, 5, and 10, respectively. Find the moments M_x and M_y about the *x*- and *y*-axes respectively, and find the center of mass of the system.

3 Find the *y*-coordinate of the center of mass of the region bounded by the curves $x = y^2 - 5$ and $x = 4y$, if the lamina's density is given by the function $\delta = y + 3$. Your solution should include a sketch of the region as well as the calculations needed to produce your answer; you may use your calculator to evaluate integrals.

4 Prove using the delta-epsilon definition of a limit that $\lim_{x \to 2} \frac{4}{x-3} = -4$. Write your proof in the correct order and include a statement of the definition of a limit.

5 Let *R* be the region defined by $2x^2 + 2y^2 + 12x - 20y - 4 \le 0$. Use the Theorem of Pappus to find the volume of the solid formed when *R* is revolved about the line $x = 6$. (No calculus is necessary.)

6 Consider the half of the hyperbola $x^2 - y^2 = 1$ for which *y* is nonnegative. Write parametric equations for this curve using *t* as the parameter, where *t* is the slope of the line tangent to the hyperbola at the point (x, y).

7 Use the distance formula in polar coordinates to find a *polar* equation for the set *P* of all points that are twice as far from $(0,6)$ as from $(0,3)$ in the Cartesian plane.

Answers

1b 11/12

2 $M_x = 10,\ M_y = 1,$ and $(\bar{x}, \bar{y}) = (1/21, 10/21)$.

3 $\bar{y} = 59/25$

5 $648\pi^2$

6 $\langle x, y\rangle = \left\langle \dfrac{t}{\sqrt{t^2-1}}, \dfrac{1}{\sqrt{t^2-1}} \right\rangle$

7 $r = 4\sin\theta$

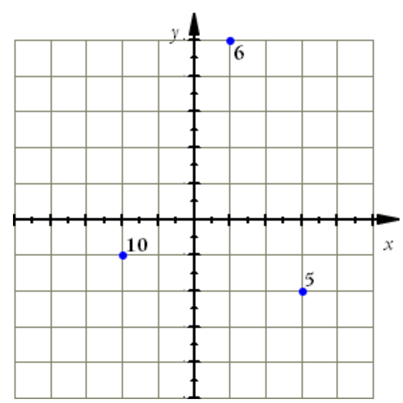
Solutions

1a Recall the identity $\cosh^2 x - \sinh^2 x = 1$; therefore $\cosh^2 x - 1 = \sinh^2 x$. The integral is therefore $\int \dfrac{\cosh x}{\sinh^2 x} dx$, which we can evaluate using u-substitution with $u = \sinh x$. Then $du = \cosh x\, dx$ and the integral is $\int \dfrac{du}{u^2}$ $= -\dfrac{1}{u} + C = -\dfrac{1}{\sinh x} + C = -\operatorname{csch} x + C.$

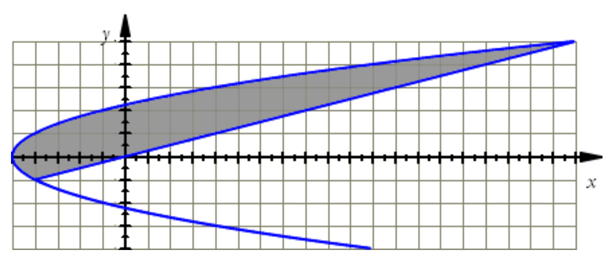
1b $\int_{\ln 2}^{\ln 5} \dfrac{\cosh x}{\cosh^2 x - 1} dx = -\operatorname{csch} x\Big]_{\ln 2}^{\ln 5}$. Now it is convenient to return to the exponential definitions of the hyperbolic trigonometric functions; in this case, recall that $\operatorname{csch} x = \dfrac{1}{\sinh x} = \dfrac{1}{\frac{1}{2}(e^x - e^{-x})}$, so we are interested in

$$-\frac{1}{\frac{1}{2}(e^x - e^{-x})}\Bigg]_{\ln 2}^{\ln 5} = -\frac{1}{\frac{1}{2}(e^{\ln 5} - e^{-\ln 5})} - \left(-\frac{1}{\frac{1}{2}(e^{\ln 2} - e^{\ln 2})}\right) = -\frac{1}{\frac{1}{2}(5 - \frac{1}{5})} + \frac{1}{\frac{1}{2}(2 - \frac{1}{2})} = -\frac{1}{\frac{1}{2}(\frac{24}{5})} + \frac{1}{\frac{1}{2}(\frac{3}{2})} = -\frac{1}{\frac{12}{5}} + \frac{1}{\frac{3}{4}}$$
$$= \frac{4}{3} - \frac{5}{12} = \frac{11}{12}.$$

2 The diagram at right may make things more convenient. The system has total mass $m = 6 + 5 + 10 = 21$. The moment about the x-axis, M_x, is given by taking the product of each particle's mass with its distance from the x-axis (note that this is the y-coordinate of each!) and adding this up for each particle. Therefore $M_x = 6(5) + 5(-2) + 10(-1) = 10$. Similarly, we use the x-coordinate in finding M_y because this represents particles' distances from the y-axis; $M_y = 6(1) + 5(3) + 10(-2) = 1$. The center of mass has coordinates $(\bar{x}, \bar{y})$ $= \left(\dfrac{M_y}{m}, \dfrac{M_x}{m}\right) = \left(\dfrac{1}{21}, \dfrac{10}{21}\right)$.



3 Again, a diagram is provided. The lamina's total mass is $m = \int \delta(y)\, dA = \int_{-1}^{5} (y+3)\left(4y - (y^2 - 5)\right) dy = 180.$ Remember that the second factor comes from the width of a small section and the dy represents an infinitesimal thickness, so their product gives a small area. To find the y-coordinate of the center of mass, $\bar{y}$, we must find the lamina's moment about the x-axis, M_x. This requires the same integral but with an additional factor of y to include the distance from the x-axis: $M_x = \int_{-1}^{5} y(y+3)\left(4y - (y^2-5)\right) dy = \dfrac{2124}{5}$.



Then $\bar{y} = \dfrac{M_x}{m} = \dfrac{2124/5}{180} = \dfrac{59}{25}$.

4 We wish to prove that $\forall\varepsilon>0\ \exists\delta>0: |x-2|<\delta \Rightarrow \left|\frac{4}{x-3}+4\right|<\varepsilon$. (If you don't remember what all the symbols mean, fix that.) We start from the end in order to find $\delta=f(\varepsilon)$. $\left|\frac{4}{x-3}+4\right|<\varepsilon$, then we combine the two parts within the absolute value signs: $\left|\frac{4}{x-3}+\frac{4(x-3)}{x-3}\right|=\left|\frac{4+4x-12}{x-3}\right|=\left|\frac{4(x-2)}{x-3}\right|<\varepsilon$. The $x-2$ factor is desirable because we will be starting from a statement involving it, but we need to eliminate a factor of $\frac{1}{x-3}$. Thus we restrict δ. A common first instinct is to try letting $\delta<1$, which produces $-1<x-2<1$. We want a factor of $x-3$, so $-2<x-3<0$, but since we want to take the reciprocal of both sides, this doesn't work because we would have to involve $1/0$. Instead, we take (the arbitrarily chosen) $\delta<2$. Now $-2<x-2<2$, or $-3<x-3<1$. Taking reciprocals and remembering to switch the direction of the comparison operators, $-\frac{1}{3}>\frac{1}{x-3}>1$. Knowing that $\frac{1}{x-3}<-\frac{1}{3}$ allows us to state $\left|4\left(-\tfrac{1}{3}\right)(x-2)\right|<\left|\frac{4(x-2)}{x-3}\right|<\varepsilon$, so $\left|\tfrac{4}{3}(x-2)\right|<\varepsilon$. Therefore $|x-2|<\tfrac{3}{4}\varepsilon$. Now we can begin with the actual proof.

Begin by stating that $\forall\varepsilon>0$ we will let $\delta=\min\{2,\tfrac{3}{4}\varepsilon\}$. Therefore $|x-2|<\delta$ has two possibilities: first, $|x-2|<2$, or $-2<x-2<2$, which gives $-3<x-3<1$. Again taking reciprocals and flipping operators, $-\frac{1}{3}>\frac{1}{x-3}>1$, so we know that $\frac{1}{x-3}<-\frac{1}{3}$. Let's just set this aside for a moment and continue with the other possibility, $|x-2|<\tfrac{3}{4}\varepsilon$. This means that $\left|\tfrac{4}{3}(x-2)\right|<\varepsilon$, or $\left|\tfrac{1}{3}(4)(x-2)\right|<\varepsilon$. We have the discretion to put in a negative sign within the absolute value signs: $\left|-\tfrac{1}{3}(4)(x-2)\right|<\varepsilon$. Using what we just "discovered", we can replace the $-1/3$ with $\frac{1}{x-3}$ because $\left|\frac{1}{x-3}(4)(x-2)\right|<\left|-\tfrac{1}{3}(4)(x-2)\right|<\varepsilon$. This gives $\left|\frac{4(x-2)}{(x-3)}\right|=\left|\frac{4}{x-3}+4\right|<\varepsilon$, which is what we wanted. We have just shown that $\forall\varepsilon>0\ \exists\delta>0: |x-2|<\delta \Rightarrow \left|\frac{4}{x-3}+4\right|<\varepsilon$, which is equivalent by definition to the statement $\lim_{x\to 2}\frac{4}{x-3}=-4$.

5 Recall that the [relevant part of the] Theorem of Pappus states that the volume of a solid of revolution generated by revolving a planar region about an external axis is the product of the area of the region with the distance its centroid travels in revolution. First, it is important to determine what exactly the region is. Begin by rewriting the region's defining inequality as $x^2+6x+y^2-10y\le 2$ and then complete the square for x and y: $x^2+6x+9+y^2-10y+25\ \le 2+9+25$, or $(x+3)^2+(y-5)^2\le 36$. This, then, is a circle with radius 6 centered at $(-3,5)$. The region's centroid is simply the circle's center, from which the distance to the line $x=6$ is 9. The centroid will travel a distance of 2π times its distance from the axis, or 18π, and the region's area is clearly $\pi r^2=6^2\pi=36\pi$. Therefore the volume of the solid is $18\pi(36\pi)=648\pi^2$.

6 We differentiate the given equation implicitly with respect to *x*, giving $2x-2y\frac{dy}{dx}=0$ and solve for $\frac{dy}{dx}$ to get $\frac{dy}{dx}=\frac{x}{y}$. The problem states that the parameter *t* is to be assigned to $\frac{dy}{dx}$, so we know $t=\frac{x}{y}$. It then remains to solve for x and y in terms of t only. We square the just-found equation to get $t^2 = \frac{x^2}{y^2}$, and use the original definition of the curve to substitute $x^2 = 1 + y^2$. Now $t^2 = \frac{1+y^2}{y^2}$, so $y^2\left(t^2 - 1\right) = 1$ and $y^2 = \frac{1}{t^2-1}$ giving $y = \frac{1}{\sqrt{t^2-1}}$. Changing the substitution to be instead $y^2 = x^2 - 1$, we also can find $t^2 = \frac{x^2}{x^2-1}$, so $x^2\left(t^2-1\right) = t^2$ and $x^2 = \frac{t^2}{t^2-1}$. This gives $x = \frac{t}{\sqrt{t^2-1}}$, so the parametric equations are $\langle x, y\rangle = \left\langle \frac{t}{\sqrt{t^2-1}}, \frac{1}{\sqrt{t^2-1}} \right\rangle$.

7 The formula for the distance d in polar coordinates between two points given by $\left(r_1, \theta_1\right)$ and $\left(r_2, \theta_2\right)$ is $d = \sqrt{r_1^{\,2} + r_2^{\,2} - 2r_1r_2\cos\left(\theta_2 - \theta_1\right)}$. You can derive this from the Law of Cosines if you have forgotten it, but it would be more advisable to memorize it. Realize that we are concerned with three points: an arbitrary (r, θ), and the two given points. These are, in polar coordinates, $(6, \pi/2)$ and $(3, \pi/2)$ respectively. Therefore according to the problem, we want to find all (r, θ) such that $2\sqrt{r^2 + 3^2 - 2r(3)\cos\left(\theta - \pi/2\right)} = \sqrt{r^2 + 6^2 - 2r(6)\cos\left(\theta - \pi/2\right)}$. It is convenient to square both sides, giving $4\left(r^2 + 9 - 6r\cos\left(\theta - \pi/2\right)\right) = r^2 + 36 - 12r\cos\left(\theta - \pi/2\right)$, which is more nicely written as $4r^2 + 36 - 24r\sin\theta = r^2 + 36 - 12r\sin\theta$, or even better as $3r^2 = 12r\sin\theta$. This can be rewritten one more time, in standard $r = f(\theta)$ form, as $r = 4\sin\theta$.