Discussion Notes: Dimensional Analysis

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1 Introduction

In Physics C, we use the "SI" system of units (yes, that's redundant). SI is an abbreviation for something in French. Other systems include the Imperial system (inch, pound, etc.) and the CGS system. The CGS system is another metric system, but it uses the centimeter and gram as its base units for length and mass, respectively, rather than the meter and the kilogram as does the SI (or MKS, meter-kilogram-second) system. We will use one CGS unit, the Gauss, in the E&M section of the course, but otherwise restrict ourselves to MKS.

2 Base Units

There are seven base units to measure seven fundamental quantities:

- length: meter [m]
- mass: kilogram [kg]
- time interval: second [s]
- electric current: ampere [A]
- luminous intensity: candela [cd]
- absolute temperature: kelvin [K]
- amount of a substance: mole [mol]

We won't worry about the last three, and the ampere won't show up until second semester.

3 Prefixes

We can modify units by orders of magnitude by putting in front of them one (and only one) of the following symbols:

- mega-, M $= 10^6$
- kilo-, k $= 10^3$
- milli-, m $= 10^{-3}$
- micro-, $\mu = 10^{-6}$
- nano-, n $= 10^{-9}$

There exists others in both directions, but we won't need them.

Note that capitalization counts, because mega- and milli- are certainly different.

4 Notation

Units can be multiplied and divided, as we're about to see (and as you're hopefully already seen). To avoid confusion, we separate multiplied and divided units; if we didn't, [ms] could be either a meter-second or a millisecond. It's acceptable to use either a space or a dot ($\cdot$) for this separation; I typically use the dot.

It's important to write units in a roman (ordinary) font rather than an italic one as is used for variables, so as to make it clear. In handwriting, it's good to surround units with [square brackets] when it might be ambiguous; for extra clarity, I often do that in typeset work too.

The people in charge of the SI (the International Committee for Weights and Measures) also require that you don't put a period after a unit, unless it ends a sentence; it's hardly a big deal, but worth mentioning. Capitalization is important, though not as much as for prefixes since ambiguity is more or less impossible: the rule is that all units have lowercase symbols *unless* they're named after a person (Ampere, Newton, Joule but meter, kilogram, lumen).

We'll talk about the divided units aspect of notation in a separate section, since it's a topic too important to get just a paragraph here.

5 Combining Units

If you want to add quantities, their units must match, period. It's nonsense to add a length and a mass.

To multiply or divide quantities, the units can be in any combination. (Dot and cross products are treated identically to scalar multiplication with regard to units.) As I mentioned, multiplied units are separated by a dot or a space, and divided units will have their own section.

If we differentiate, we divide the units: $\mathbf{v} = \dfrac{d\mathbf{x}}{dt}$, and since $d\mathbf{x}$ has units of [m] while dt has units of [s], we divide the units and the result, $\mathbf{v}$, has units of $\left[\mathrm{m} \cdot \mathrm{s}^{-1}\right]$.

In the case of integrating, we multiply the units: since $\mathbf{x} = \int \mathbf{v}\, dt$ and $\mathbf{v}$ has units of $\left[\mathrm{m} \cdot \mathrm{s}^{-1}\right]$ while dt has units of [s], we multiply the two. That tells us that $\mathbf{x}$ has units of [m], which we know is correct.

Often combinations of units are given names: the newton, for example, measures force and is equivalent to

$$[\mathrm{N}] = \left[\mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{s}^{-2}\right]$$

Other frequently-seen combinations of units include the joule, the watt, and the volt. Some common combinations of units don't have special names: momentum $\mathbf{p}$ is measured in $\left[\mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{s}^{-1}\right]$. (Mr. Waechtler will allow his students to refer to this quantity as the Waechtler, symbolized $[\ddot{\mathrm{W}}]$ – and be sure to use the umlaut.)

6 How to Write Divided Units

There are three options: write out a fraction, use parentheses, or use negative exponents. I'm a big fan of the last option, and have been using it throughout this text.

Here's why you don't just write something like $\left[\mathrm{kg} \cdot \mathrm{m}/\mathrm{A}^2 \cdot \mathrm{s}^2\right]$ (the units for permeability, μ): the slash is ambiguous. The rules of order of operations tell us that the seconds aren't being divided by, but in truth they *are*.

You can use parentheses:

$$\left[\mathrm{kg} \cdot \mathrm{m} \,/\, (\mathrm{A}^2 \cdot \mathrm{s}^2)\right]$$

which is ugly but permissible.

You can write it as a fraction:

$$\left[\frac{\mathrm{kg} \cdot \mathrm{m}}{\mathrm{A}^2 \cdot \mathrm{s}^2}\right]$$

which is fine. In fact, it's the clearest way. However, if you try to fit it within a line of text, it looks like this: $\frac{\mathrm{kg}\cdot\mathrm{m}}{\mathrm{A}^2\cdot\mathrm{s}^2}$. Since that's a bit hard to read, we have the next option.

You can use negative exponents. If we have a^2/b^2, we can equivalently write $a^2 b^{-2}$. In the same vein, the units we've been discussing above might as well be written as

$$\left[\mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{A}^{-2} \cdot \mathrm{s}^{-2}\right]$$

which is my favorite solution, since it eliminates any ambiguity, is less ugly than parentheses, and fits conveniently on one line.

7 What Not to Do

Don't make unit divisions ambiguous. We've already covered that.

More importantly, don't confuse units that are dimensionally equivalent but conceptually different. Work may be expressed (for a constant force. . . the discussion's coming) as

$$W = \mathbf{F} \cdot \Delta\mathbf{s}$$

and since the units of $\mathbf{F}$ are newtons and of $\Delta\mathbf{s}$, meters, the units of work W are newton-meters. In this case, there's a name for that composite unit: the joule, [J].

However, torque may be expressed as $\tau = \mathbf{r} \times \mathbf{F}$, where again we have a quantity measured in newtons multiplied by a quantity measured in meters. But with torque, the length in question is *perpendicular* to the force, and so the concept of torque is completely different from the concept of work. Therefore, we simply refer to torque's units as $[\mathrm{N} \cdot \mathrm{m}]$.

8 Multiple-Choice Strategy

Suppose a multiple choice question presents a scenario and you are asked to determine the centripetal force in terms of m, v, and r. To save me from the scratchwork, there are three choices rather than the usual five:

- $\|\mathbf{F}_c\| = \dfrac{mv}{r}$
- $\|\mathbf{F}_c\| = \dfrac{mv^2}{r}$
- $\|\mathbf{F}_c\| = \dfrac{mv}{r^2}$

If you have no idea how to proceed, work with the units. First notice that since we're looking for a force, we want units of $\left[\mathrm{N} = \mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{s}^{-2}\right]$. And since the exponents of v and r don't occur in the same combination in any of those choices, only one will have the right units (assuming any of them do).

Working out the units for the first choice, we get $\left[\mathrm{kg} \cdot \mathrm{s}^{-1}\right]$; for the second, $\left[\mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{s}^{-2}\right]$; and for the third, $\left[\mathrm{kg} \cdot \mathrm{m}^{-1} \cdot \mathrm{s}^{-2}\right]$. Only the second of these fits our criterion, so it must be the right answer. . . and you got it without knowing any actual physics. Congratulations.